GRAVITATIONAL COLLAPSE OF
MAGNETIC NEUTRON STARS*

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Abstract. Pulsars are presently believed to be rotating neutron stars with frozen-in magnetic fields. Because of the high density of neutron stars, general relativistic effects are important since they effect both the structure and stability of such stars. Besides this, the magnetic field outside the star is also affected. Instead of falling off as $r^{(2+1)}$ as in flat space, it is shown that each magnetic multipole varies as a hypergeometric function of radius. A closed form of these hypergeometric functions is given in terms of Legendre functions of the second kind. If the mass of a neutron star exceeds about $2.4 \, m_\odot$, the star becomes unstable and collapses. For a quasistatically collapsing body, it is shown that the magnetic field seen by a distant observer vanishes as the radius approaches the gravitational radius.

1. Introduction

The recent discovery of pulsars (Hewish et al., 1968) has generated renewed interest in neutron stars, in particular, neutron stars with frozen-in magnetic fields (Goldreich and Julian, 1969; Gunn and Ostriker, 1969a; Canuto and Chiu, 1968). It is presently believed that pulsars are rotating neutron stars (Gold, 1968) which are being slowed down by an off-axis magnetic field (Ginzburg and Ozernoi, 1965).

It has further been suggested that the rotational energy of the pulsar in the Crab Nebulae is the energy source of the nebulae (Wheeler, 1967; Finzi and Wolf, 1969). If this is true, then the observed energy emitted by the Crab will give a lower limit on the rate of loss of rotational energy. This rate can be computed since both the period and rate of change of the period of the pulsars have been measured (Craft et al., 1969; Cocke et al., 1969). When the calculations were carried out (Cohen and Cameron, 1969), taking general relativistic contributions to the rotational energy into account, it was found that the Crab pulsar must have a minimum mass of $\sim 0.4 \, m_\odot$. These results lend support to the now generally accepted idea that pulsars are rotating neutron stars.

Such neutron stars can be quite dense. Recent neutron star calculations (Cohen et al., 1970; Cohen and Cameron, 1970), based on an improved equation of state, have given stable models with radius ($\sim 12 \, \text{km}$) as small as 1.6 times the gravitational

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radius. The spatial curvature in the neighborhood of such a star is not negligible and can have an effect not only on the structure and stability of the star itself but also on the magnetic fields that may exist outside the star.

In this paper, the magnetic field exterior to a fully relativistic nonrotating neutron star is considered. The main interest will be to find the effect of the neutron star on the electromagnetic field rather than visa versa. Because of general relativistic effects, the magnetic fields can be quite different from those in flat space. For example, in flat space, each axially-symmetric magnetic multipole field of order $l$ varies with radius as $r^{-2l-1}$; whereas in the curved space exterior to a neutron star, we will show that each magnetic multipole varies as a hypergeometric function of radius. These hypergeometric functions will be expressed in terms of Legendre functions of the second kind which are known in closed form.

Using these analytic solutions, we will treat the adiabatic collapse of a body with an arbitrary axially symmetric multipole field. Ginzburg and Ozernoi (1965) have discussed the quasistatic collapse of a star with a frozen-in dipole magnetic field. However, they used as a model for the interior of the star the Oppenheimer and Snyder (1939) interior solution. Since the Oppenheimer-Snyder solution describes a star in free collapse, it is perhaps not the best model of a quasistatically collapsing star as Ginzburg and Ozernoi pointed out. We will show that the result of Ginzburg and Ozernoi, namely that the exterior dipole magnetic field goes to zero as the radius of the star approaches its Schwarzschild radius, can be extended to arbitrarily high multipoles for arbitrary interior solutions consistent with assumptions of spherical symmetry and quasistatic collapse.

2. Curved Space Maxwell Equations

Since the main interest of this paper is the effect of the neutron star on the electromagnetic field rather than visa versa, we assume that the metric is the same as that for a neutron star with no magnetic field (if the magnetic field is sufficiently weak). The gravitational field exterior to any spherical body is given by the Schwarzschild metric

$$ds^2 = -e^\nu dt^2 + e^\lambda dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2,$$

where

$$e^\nu = e^{-\lambda} = 1 - \frac{2Gm}{rc^2}.$$

Here $m$ is the mass of the star, $G$ is the gravitational constant, $c$ is the speed of light, and $r$ is the radius. The radial parameter $r$ has the property that the area of a sphere of radius $r$ is $4\pi r^2$. In the sequel, we shall use $m$ in place of the quantity $Gm/c^2$.

Maxwell's equations for a given gravitational field characterized by the metric $g_{\mu\nu}$, can be written in the form

$$F_{\mu\nu} = 0,$$  

and

$$F_{\mu\nu, \alpha} + F_{\alpha\mu, \nu} + F_{\nu\alpha, \mu} = 0.$$