POTENTIAL ENERGY OF CONTINUOUS AND DISCRETE DISTRIBUTIONS OF MATTER FROM THE POINT OF VIEW OF THE APPLICATION OF THE VIRIAL THEOREM*

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Abstract. The potential energy of clusters of stars in which the distribution of matter is taken to be continuous is compared with that of static model clusters in which the distribution of matter is discrete, the comparison being made from the point of view of applying the virial theorem to estimate the masses of the clusters. There is good agreement on the average between the two cases as long as the stellar distribution is random. Systematic differences occur whenever there is any departure from randomness. However, reduction of the mass of a cluster as estimated by means of the virial theorem by even as much as a factor of 2 on the average would seem to require even greater departures from randomness in the stellar distribution than are considered here. As might be expected there are sometimes very large fluctuations in the potential energy from one cluster to the next in the discrete case.

1. Introduction

Of the two quantities needed to estimate the mass of a cluster of stars or galaxies by means of the virial theorem only one, the dispersion in radial velocities, is obtained from observations. The other, the potential energy of the cluster, depends on some assumed model for the distribution of mass within the cluster and it is customary (see for example Chandrasekhar, 1960, pp. 200–201) to represent the potential by means of a characteristic length, $2L$ (the factor 2 is traditional), such that for a cluster of $N$ stars each of mass $m$,

$$\frac{M^2}{2L} = m^2 \sum_{1 \leq i < j \leq N} \frac{1}{r_{ij}},$$

(1)

where $M$ is the mass of the entire cluster and $r_{ij}$ is the distance between the $i$th and $j$th stars. The virial theorem holds that the mass of the cluster is proportional both to the observed dispersion in velocities and to the length $L$. Dibai et al. (1961) derived an expression for $L$ for various continuous distributions of matter and it is the purpose here to repeat their calculations for discrete model clusters of from 50 to $10^3$ stars, the coordinates of the stars being chosen in some sense at random. Schwarzschild (1954), also, has obtained an expression for $L$ but in terms of counts of stars (galaxies) in parallel strips across the face of the cluster but this too tacitly assumes a continuous distribution of matter within the cluster.

Clearly one is concerned here with the adequacy of the representation of a discrete

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distribution of matter by a continuous distribution and while this might be considered relatively unimportant in general it is worth examining in the present context, if only to get quantitative estimates for such qualitative conjectures as the possible effects of close neighbours, because of the old and well-known problems which arise when the virial theorem is applied to clusters of galaxies (Neyman et al., 1961).

Since \( r_{ij} \) appears as a reciprocal in Equation 1 it is formally and strictly true – but also trivial – that the occurrence of even a single arbitrarily small mutual separation can make the corresponding value of \( L \) arbitrarily small as well. More realistically the behaviour of \( L \) is related to the statistics of the distribution of points in space. Since the object of this paper is to present numbers and, since such numbers have often only as much meaning as the method by which they were obtained, the construction of the model clusters used here is described in rather more detail than has been customary in the literature hitherto.

2. Construction of the Models

Each cluster was assumed to have a radius of 1 and to contain \( N \) stars of equal mass \( m \). Equation 1 can thus be rearranged so that

\[
L = \frac{N^2}{2 \sum \frac{1}{r_{ij}}}, \tag{2}
\]

where \( r_{ij} \) is measured in units of the cluster's radius. The principal tool was a machine supplied programme for the generation of pseudo-random numbers, uniformly distributed in the interval \( 0 \rightarrow 1 \), by the standard technique of 'multiplicative congruences' (Hamming, 1962, pp. 384–388; Dobell and Hull, 1962) which was used to obtain the coordinates of the stars in the clusters. \( L \) can be calculated directly from Equation 2 once a cluster of \( N \) stars has been constructed.

A. CONSTRUCTION OF AN HOMOGENEOUS CLUSTER

An homogeneous cluster of stars is one with the same density on the average throughout. The longitude \( (\theta_i) \) and co-latitude \( (\varphi_i) \) of the \( i \)th star were obtained from the relations

\[
\theta_i = 2\pi P_i, \tag{3}
\]
\[
\varphi_i = \cos^{-1} (2Q_i - 1), \tag{4}
\]

where \( P_i \) and \( Q_i \) were chosen 'at random' by the machine (IBM 360/65). As an illustration of the resulting distribution for a cluster of \( 10^5 \) stars the surface of a sphere was divided (by latitude and longitude) into sixteen portions of equal area and the number of points \((\theta, \varphi)\) falling within each division given in Table I. The third coordinate, the distance of the star from the centre of the cluster \( (r_i) \), was obtained from the relation

\[
r_i = R_i^{1/3}, \tag{5}
\]