II. Main-Sequence Stars

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Abstract. It was shown in a previous paper (Smith, 1976) that the method of strained coordinates may be usefully employed in the determination of the structure of rotating polytropes. In the present work this idea is extended to Main-Sequence stars with conservative centrifugal fields. The structure variables, pressure, density and temperature are considered pure functions of an auxiliary coordinate $s$ (the strained coordinate) and the governing equations written in a form that closely resembles the structure equations for spherical stars but with correction factors that are functions of $s$. A systematic, order-by-order derivation of these factors is outlined and applied in detail to a Cowling-model star in uniform rotation. The technique can be extended beyond first order and external boundary conditions are applied, as they should be, at the true surface of the star. Roche approximations are not needed.

1. Introduction

In a previous paper (Smith, 1976, hereafter referred to as Paper I) the structure of rotating polytropes was calculated correct to third order by utilization of the method of strained coordinates or PLK method, see e.g. Van Dyke (1964). In compliance with this procedure we introduced a strained radial coordinate $s$ via the transformation

$$r = r(s, \lambda), \quad \mu = \lambda,$$

with $r$ and $\mu (= \cos \theta)$ referring to spherical polar coordinates, so that the surface of the rotating polytrope mapped onto the fixed surface

$$s = s_0,$$

which is spherical in the strained space.

In physical space this surface is (i) unknown but (ii) definitely non-spherical, and matching the gravitational potential across it to a finite solution of Laplace's equation becomes a formidable task. This has led several authors, such as Monaghan and Roxburgh (1965), Martin (1970), James (1964), Ostriker and Mark (1968), to perform the matching procedure over an arbitrary sphere rather than at the true surface. Although the errors involved in this practice are undoubtedly small, depending upon the degree of central condensation of the model, this still remains a rather unsatisfactory state of affairs, for the approximation cannot be improved. Sooner or later one is faced with the complexities of matching across a non-spherical interface; this has not been attempted.

The use of strained coordinates at once resolves the dilemma. One merely transforms the gravitational potential $\Phi$, to $s, \lambda$ space and matches across the known, spherical interface $s = s_0$. This procedure is exact, consistent with the order of approximation.
involved in calculating $\Phi$, and improves at higher orders. Just as the surface of the rotating configuration is unknown in physical space, so the functional form of the transformation (1.1) is initially undefined, but is determined step-by-step as the solution progresses. A zero-order (non-rotating) approximation to the structure of the polytrope enables the straining to be calculated correct to first order. This is then used to derive the first-order structure and thence the second-order straining, and so on.

The method is particularly (but not exclusively) useful for rotation laws of the form

$$\Omega = \Omega(\tilde{\omega}),$$

(1.3)

where $\tilde{\omega}$ is the cylindrical radial coordinate and $\Omega$ the angular velocity, for it is then natural to associate the strained variable $s$ with the total (gravitational plus centrifugal) potential $\Psi$ according to:

$$\Psi = \Psi(s).$$

(1.4)

For polytropes the surface is one of constant $\Psi$ and, hence, also one of constant $s$, in agreement with (1.2).

In the present paper we generalize the approach of Paper I for application to Main-Sequence stars with rotation laws of the form (1.3). With the strained coordinate $s$ again characterizing equipotential surfaces, it is possible to write the governing equations as a system of ordinary differential equations with $s$ as independent variable that are qualitatively similar to those pertaining to spherical stars but with correction factors that are functions of the transformation (1.1). The derivation of these equations is exact and formally equivalent to that of Papaloizou and Whelan (1973) and Kippenhahn and Thomas (1970), but in appearance more closely resembles the formulation of Faulkner et al. (1968) to which it reduces at first order. To illustrate the technique a simple numerical model is considered in some detail and second order results presented.

It should be emphasized that several assumptions have been made in writing down the fundamental equations of a rotating star in the form that appears in the next section. The redistribution of angular momentum by the large-scale meridian circulation is ignored. Rather it is assumed that the angular velocity is constant on cylinders from which it follows that the pressure, density and temperature are constant on equipotential surfaces (Von Zeipel, 1924). Also in the surface layers the omission of the inertia of the circulation – see Smith (1977) – as well as the use of the point equation of radiative equilibrium – see Smith (1966, 1970), Osaki, (1966, 1972) – both become suspect. For a discussion of these and other aspects of rotating stars the reader is referred to excellent review articles by Mestel (1965), Strittmatter (1969), Roxburgh (1970) and Fricke and Kippenhahn (1972).

2. Fundamental Equations

The governing equations for a star rotating with angular velocity $\Omega$ are, in the usual notations,