Abstract. It is reiterated that any suggestion of the existence of a 'third integral' is at variance with Poincaré's theorem on the non-existence of such integrals. Even in a purely numerical approach no form of a new 'integral' can be constructed that is valid in every domain of the phase space; and it is devoid of meaning to use as a 'third integral' different forms of functions in various cases.

In the first issue of this journal (see Goudas, 1968) this author presented evidence, both theoretical and numerical, according to which the 'third integral' for the galaxy given by Contopoulos (1963) when applied in any domain of the phase space, should either diverge or be a function of the Hamiltonian and as a result is not a new integral. In the preceding pages, Contopoulos and Barbanis take issue with this view and present one theoretical argument and two numerical examples that supposedly contradict the dependence of the third integral and the Hamiltonian. However, as the subsequent discussion will show, the above paper does not offer any basis for changes in the conclusions drawn by this author in his original paper.

To start with the theoretical argument, it must be pointed out that Contopoulos and Barbanis present nothing new but repeat the words of Poincaré, who based the proof of the non-existence of other single-valued integrals on the fact that the assumption of existence of a new integral leads to the antinomy that \( \phi_0 \) must be a function of the Hamiltonian \( H_0 \). We shall quote here the book of E. T. Whittaker Analytical Dynamics of Particles and Rigid Bodies, in which a very clear exposition of the respective theorem of Poincaré is given.

Indeed Whittaker writes:

"(iii). \( \phi_0 \) cannot be a function of the \( H_0 \)" (page 381). He continues to give the proof of this statement, and later (page 383) presents and gives the proof of the main part of the non-existence theorem of Poincaré writing:

"(v). Proof that the existence of a one-valued integral is inconsistent with the result (iii)."

As far as the numerical examples are concerned, we should first point out that four out of the six orbits computed by Contopoulos and Barbanis aim at proving, perhaps more accurately, my thesis; namely, that the third integral and the Hamiltonian have collinear gradients along periodic solutions. Yet, in reality, the orbits used to test the above thesis are only approximately periodic, or, lie close to precise periodic solutions of known properties, and in this way, the collinearity of the gradients is shown to be valid in the vicinity of periodic orbits. The possibility that the conjecture of Poincaré and Birkhoff about the density of periodic solutions may be correct - so that state-
ments such as "a little further away" from periodic orbits make no sense – suggested to this author that all single-valued and differentiable functions of position in phase space that are conserved along solutions of the equations of motion must be functions of the Hamiltonian. This view is as valid, or incorrect, as the said conjecture of Poincaré and Birkhoff.

But Contopoulos and Barbanis avoid this difficulty by stating that their integral "\( \phi \) can take an infinity of forms" and still clearer that "one cannot find a form which is valid everywhere". This author agrees with this view and wishes to add that the forms that the third integral can take are as many as the solutions of the equations of motion, i.e. to each solution corresponds one expression of \( \phi \) and vice-versa, but refuses to give to this infinite set of functions the title 'integral'.

In spite of their view that "one cannot find a form of \( \phi \) which is valid everywhere, or, at least, in some domain of the phase space", the above authors applied this integral for a domain of finite dimensions on various occasions one of which is the paper of the preceding pages. The form of \( \phi \) that was used along the orbits of Tables II and III that supposedly contradict the thesis of dependence of \( \phi \) on \( H \), is not a suitable one. This must be so because as close as we like to each of the orbits of Tables II and III, we can find a periodic solution along which the third integral and the Hamiltonian will have collinear gradients. For this periodic solution the third integral will admittedly have a different form and, of course, a different gradient. This new gradient will be collinear to the gradient of the Hamiltonian. An example of this is given by Contopoulos and Barbanis in Tables VI and VII. In Table VII it is shown that the form of the third integral corresponding to another orbit does not fulfil the necessary condition of having collinear gradient with the gradient of the Hamiltonian. On the contrary, the two gradients make very large angles. This, however, the above authors corrected very efficiently by changing the form of the third integral. The new form has the required property, as is demonstrated in Table VI. They can do likewise for the orbits of Tables II and III, since these are also numerically as periodic as is that of Table VI, although their period may be very large.

It can be argued, however, that there is no reason to use another form of the third integral for the orbits listed in Tables II and III on account of the fact that the constant is conserved reasonably well. The conservation of the constant of the third integral in these two cases is due to the small size of the orbits which make the non-linear terms contribute very little to the constant thus leaving the linear terms of the equations of motion to manifest the existence of another independent integral which indeed exists when the non-linear terms vanish. If one tries to do the same for larger orbits the conservation of the constant is worse and new forms for the integral are necessary as is shown in Table VI and VII of Contopoulos and Barbanis.

To summarize what seems to be the entire story and meaning of the third integral the following points must be stressed.

First, we agree with Contopoulos and Barbanis that no complete expression of \( \phi \) can be found so that it will be valid for all motions within any finite domain of phase