THE GENERAL DUST SOLUTIONS IN THE 
BRANS-DICKE THEORY 

(Letter to the Editor) 

DIETER LORENZ-PETZOLD 
Fakultät für Physik, Universität Konstanz, F.R.G. 

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Abstract. We discuss the Brans-Dicke field equations for the non-flat \(k^2 = 1\) Friedmann-Robertson-Walker models filled with dust. It is shown that the problem of constructing the general dust solutions can be reduced to the problem of solving a first-order differential equation. 

1. Introduction 

In previously published papers (Lorenz-Petzold, 1983a, b, 1984) we presented new exact solutions of the Brans-Dicke theory (BDT) of gravitation for spatially homogeneous models of the Friedmann-Robertson-Walker-Bianchi-Kantowski-Sachs space-times. Exact solutions have been given in the vacuum case as well as for FRW-models filled with stiff or radiation matter. In this paper we consider the non-flat \(k^2 = 1\) FRW-models filled with dust. The flat \(k = 0\) general BDT–FRW dust solution has been first given by Gurevich et al. (1973). The special \(k = 0\) solution has been previously given by Brans and Dicke (1961). However, no general BDT–FRW \(k^2 = 1\) dust solutions have been found until now. 

It is interesting to notice that there are no FRW \(k^2 = 1\) vacuum solutions in the general theory of relativity (GRT). This is in contrast with the BDT, where such solutions have been found very recently (Cerveró and Estévez, 1983; Chauvet, 1983; Lorenz-Petzold, 1983a). On the other side, the general FRW \(k^2 = 1\) dust solutions are well known (Kramer et al., 1980), but the corresponding general BDT-solutions are unknown. It therefore seems to be of interest to discuss the BDT–FRW \(k^2 = 1\) dust problem as far as possible. It is shown that the problem of finding the general dust solution can be reduced to a first-order differential equation. After solving this equation the most general BDT–FRW \(k^2 = 1\) dust solution would arise. 

2. Field Equations and Solutions 

The BDT–FRW \((k^2 = 1)\) field equations to be solved are: 

\[
\begin{align*}
(\ln R) + 3 (\ln R)^2 + 2k/R^2 + (\ln \phi) (\ln R)^2 = \varepsilon [1 + \omega (2 - \gamma)] / (3 + 2\omega)\phi, \\
(\ln R)^2 + k/R^2 + (\ln \phi) (\ln R)^2 - (\omega/6) (\ln \phi)^2 = \varepsilon / 3\phi,
\end{align*}
\]

\(\psi_1, \psi_2, \psi_3, \psi_4\)
\( (R^3 \phi) = \varepsilon (4 - 3\gamma) R^3 / (3 + 2\omega), \)  

where \( R = R(t), \phi = \phi(t) \) and \( (\cdot)' = d/dt. \) The perfect fluid matter is described by the equation of state

\[ p = (\gamma - 1) \epsilon, \quad 1 \leq \gamma \leq 2, \]

where \( \epsilon \) and \( p \) are, respectively, the density and pressure of the fluid. The perfect fluid obeys the conservation equation

\[ \epsilon = m R^{-3\gamma}, \quad m = \text{const.} \]

The cases \( m = 0 \) (vacuum), \( \gamma = 2 \) (stiff matter), and \( \gamma = \frac{4}{3} \) (radiation) have been already discussed by us (Lorenz-Petzold, 1983a, c). We now consider the case \( \gamma = 1 \) (dust). From Equation (3) we obtain the first integral

\[ R^3 \phi = \left( \frac{m}{3 + 2\omega} \right) t + a, \]

where \( a = \text{const.} \), which may be set equal zero without loss of generality. Substitution of Equation (6) into the Equations (1) and (2) yields

\[ \begin{align*}
(ln R)' + 3(ln R)^2 + 2k/R^2 + [(ln R)' - (1 + \omega)t^{-1}] (ln \phi)' &= 0, \\
(ln R)^2 + k/R^2 + [(ln R)' - \left( \frac{1}{3} \right) (3 + 2\omega)t^{-1}] (ln \phi)' - (\omega/6) (ln \phi)^2 &= 0.
\end{align*} \]

The linear combination of these equations can be rewritten as

\[ (ln R)' + (ln R)^2 - [(ln R)' - \left( \frac{1}{3} \right) (3 + \omega)t^{-1}] (ln \phi)' + (\omega/3) (ln \phi)^2 = 0. \]

It is now an easy matter of calculation to solve Equation (7a) for \( (\ln \phi)' \) and inserting in Equation (8) results into the following second-order differential equation for \( R = R(t): \)

\[ \begin{align*}
3R\ddot{R} [\ddot{R} - (1 + \omega)Rt^{-1}]^2 + [R\dddot{R} + 2\dddot{R}^2 + 2k] [3\dddot{R} - (3 + \omega)Rt^{-1}] \times \\
\times [\dddot{R} - (1 + \omega)Rt^{-1}] + \omega [R\dddot{R} + 2\dddot{R}^2 + 2k]^2 &= 0.
\end{align*} \]

After solving Equation (9) for \( R = R(t) \) the general FRW \( (k^2 = 1) \) dust solution would be completed by solving Equation (6) for \( \phi = \phi(t). \) However, we first consider the special case \( \dddot{R} = 0. \) The corresponding solutions are given by

\[ R = at, \quad \phi = -\left( \frac{m}{(3 + 2\omega)a^3} \right) t^{-1}, \quad a = \left( -2k/(2 + \omega) \right)^{1/2}. \]