FAMILIES OF SYMMETRIC-PERIODIC ORBITS IN THE ELLIPTIC THREE-DIMENSIONAL RESTRICTED THREE-BODY PROBLEM

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Abstract. With an orbit of the three-dimensional circular problem as a starting point, we have calculated families of symmetric-periodic orbits in the three-dimensional elliptic problem with a variation of the mass ratio $\mu$ and the eccentricity $e$. Afterwards, we have studied their evolution and stability.

1. Introduction

It is well known that, in the circular problem of three bodies, we can have periodic orbits with any value of the semi-period $T$. Thus, we can have the meaning of the family, in the space of the initial conditions, as a function of the period. This, however, cannot take place in the elliptic problem because the period can only be a multiple integer of $2\pi$. In the case of the elliptic problem we can have the meaning of the family by taking as a parameter the eccentricity $e$, or the mass ratio $\mu$ of the primaries, in the space of the initial conditions.

Furthermore, it is also known that the elliptic periodic orbits approach the real motions of the celestial bodies more closely than the circular ones. For this reason, and also because the circular orbits have been extensively studied, we decided to study the evolution and behaviour of a circular orbit with a period $2\pi$, in the phase space $(x, y, z, \dot{x}, \dot{y}, \dot{z})$, introducing the eccentricity of the primaries, changing it, as well as changing the mass ratio. Thus, with the previous members of the family we decided to create families of periodic orbits, and at the same time study their stability.

The present study deals with the numerical investigation of the periodic orbits of the three-dimensional elliptic problem. This is achieved by extending the known periodic orbits of the circular problem.

Of previous investigators of this and similar subjects, Hunter (1967) investigated the motions of the satellites of Jupiter and of the asteroids in the Sun–Jupiter system. The orbits of the satellites are taken to be examples of the elliptical restricted, three-dimensional, three-body problem. Broucke (1969) has worked systematically on the plane-elliptic restricted three-body problem and he has found many families of periodic orbits and their stability. Finally, Katsiaris (1972) and Macris et al. (1975) have given some periodic-symmetric solutions of the three-dimensional elliptic orbits and their stability.

We use the equations of motion in the vibrating and rotating coordinate system (Kopal and Lyttleton, 1963), and we take the true anomaly $v$ as an independent variable.
Hence, we set

\[
\begin{align*}
\frac{dx_1}{dv} &= x_4 = f_1, & \frac{dx_2}{dv} &= x_5 = f_2, & \frac{dx_3}{dv} &= x_6 = f_3, \\
\frac{dx_4}{dv} &= 2x_5 + \frac{r}{p} \frac{\partial U}{\partial x_1} = f_4, & \frac{dx_5}{dv} &= -2x_4 + \frac{r}{p} \frac{\partial U}{\partial x_2} = f_5, \\
\frac{dx_6}{dv} &= -x_3 + \frac{\partial U}{\partial x_3} = f_6,
\end{align*}
\]

(1)

where \(x_1, x_2, x_3\) are the coordinates and \(x_4, x_5, x_6\) the corresponding momenta. Also the potential function is defined as

\[U = (1 - \mu) \left( \frac{1}{r_1} \frac{r_1^2}{2} + \mu \left( \frac{1}{r_2} \frac{r_2^2}{2} \right) \right),\]

(2)

where

\[r_1^2 = (x_1 + \mu)^2 + x_2^2 + x_3^2, \quad r_2^2 = (x_1 + \mu - 1)^2 + x_2^2 + x_3^2, \quad p = 1 - e^2,\]

and \(m_1, m_2\) the masses of the primaries.

2. The Symmetric-Periodic Orbits

By use of Equations (1) and the potential function (2), we can see immediately that if there are any symmetrical-periodic orbits then the coordinates of two symmetrical points from these orbits will be equal to or opposite, according to the symmetry.

The function \(U\) is of the form \(U = U(x_1, x_2^2, x_3^2)\). Therefore, it will be symmetric with respect to the \(Ox_1\)-axis and to the \(Ox_1x_2\)-plane and \(Ox_1x_3\)-plane.

By starting with the circular problem at the times \(t\) and \(-t\) it follows that

\[
\begin{align*}
x_i(t) &= x_i(-t), & i &= 1, 5, 6, \\
x_i(t) &= -x_i(-t), & i &= 2, 3, 4.
\end{align*}
\]

(3)

As we can observe from Equations (1) (where \(v = t\)), the differentials of \(x_1 \ldots x_6\) are symmetric. Therefore, the relations (3) are maintained throughout the whole orbit. This means that the orbit is symmetric with respect to the \(Ox_1\)-axis.

The relations (4) and (5) present symmetry with respect to the \(Ox_1x_3\)-plane (\(T = \) half-period): namely,

\[
\begin{align*}
x_i(t) &= x_i(T - t), & i &= 1, 3, 5, & j &= 2, 4, 6, \\
x_j(t) &= -x_j(T - t),
\end{align*}
\]

(4)