CONFORMALLY FLAT STATIC SPACE-TIME IN THE GENERAL SCALAR-TENSOR THEORY OF GRAVITATION

(Letter to the Editor)

K. SHANTHI
Department of Applied Mathematics, Andhra University, Waltair, India

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Abstract. Explicit vacuum field equations in the general scalar-tensor theory of gravitation proposed by Nordtvedt are obtained with the aid of the most general conformally flat spherically-symmetric static space-time. It is shown that the most general conformally flat spherically-symmetric static solution of Nordtvedt–Barker vacuum field equations is simply the empty flat space-time of general relativity.

1. Introduction

Nordtvedt (1970) proposed a general class of scalar-tensor gravitational theories in which the parameter \( \omega \) of the Brans–Dicke (1961) scalar-tensor theory is allowed to be an arbitrary function of the scalar field \( \phi \). The general class of scalar-tensor theories includes the Jordan (1959) and Brans–Dicke theories as special cases. Barker (1978) proposed a special case of Nordtvedt’s general class of scalar-tensor theories where the Newtonian gravitational constant \( G \) does not vary with time in the homogeneous-cosmological situation and arguments in favour of this theory were put forward.

Reddy (1979) and Rao and Reddy (1982) discussed static conformally flat solutions in the Brans–Dicke and Nordtvedt–Barker scalar-tensor theories. Later, Banerjee and Santos (1981) pointed out that their procedure is not general although the final results obtained by them were by chance, quite general and, hence, they have presented therein the most general conformally flat solutions in the Brans–Dicke theory. In this paper it is shown that the most general conformally flat static vacuum solution in the Nordtvedt–Barker scalar-tensor theory is simply the empty flat space-time of general relativity.

2. Metric and Field Equations

The Nordtvedt (1970) field equations are

\[
(3 + 2\omega) \nabla^2 \phi = 8\pi T - (d\omega/d\phi) \phi_{,k} \phi^{,k}
\]

and

\[
R_{ij} - \frac{1}{2} g_{ij} R = -8\pi \phi^{-1} T_{ij} - \omega \phi^{-2} (\phi_{,i} \phi_{,j} - \frac{1}{2} g_{ij} \phi_{,k} \phi^{,k}) - \phi^{-1} (\phi_{,i} - g_{ij} \nabla \phi) ,
\]

\( i,j = 1,2 \)
where the function \( \omega(\phi) \) is an arbitrary (positive definite) function of the scalar field \( \phi \). \( T_{ij} \) is the stress energy tensor of the matter, \( T = T^k_k \), and comma and semi-colon denote partial and covariant derivatives, respectively. Special choices for \( \omega(\phi) \) yield the theories of Jordan (1959), Brans and Dicke (1961), and Barker (1978).

We consider the metric in the form
\[
ds^2 = e^{2z} [-(ar^2 + b)^2 dt^2 + dr^2 + r^2 d\Omega^2],
\]
where \( z \) is a function of \( r \) alone and \( a \) and \( b \) are constants. It has been shown by Banerjee and Santos (1981) that Equation (2) represents the most general static conformally flat spherically-symmetric metric in isotropic coordinates. The spherical symmetry assumed implies that the scalar field \( \phi \) shares the same symmetry.

If we take \( \phi \) as a function of \( r \) only and by use of Equation (2) in (1) the explicit vacuum field equations for the most general scalar-tensor theory can be written as
\[
3z^2 + \frac{4z}{r} + \frac{4a(1 + rz')}{(ar^2 + b)} - \frac{\omega}{2} \left( \frac{\phi'}{\phi} \right)^2 + \frac{\phi'}{\phi} \left( \frac{2}{r} + \frac{2ar}{ar^2 + b} + 3z' \right) = 0,
\]
\[
2z'' + z'^2 + \frac{2z'}{r} + \frac{4a(1 + rz')}{(ar^2 + b)} + \frac{\omega}{2} \left( \frac{\phi'^2}{\phi} \right)^2 + \frac{\phi''}{\phi} + \frac{\phi'}{\phi} \left( \frac{1}{r} + \frac{2ar}{ar^2 + b} + z' \right) = 0,
\]
\[
2z'' + z'^2 + \frac{4z'}{r} + \frac{\omega}{2} \left( \frac{\phi'^2}{\phi} \right)^2 + \frac{\phi''}{\phi} + \frac{\phi'}{\phi} \left( \frac{2}{r} + z' \right) = 0,
\]
\[
(2\omega + 3) \left[ \phi'' + \phi' \left( 2z' + \frac{2}{r} + \frac{2ar}{ar^2 + b} \right) \right] = -\frac{d\omega}{d\phi} \phi'^2;
\]
where a prime superscript indicates differentiation with respect to \( r \).

3. Solutions of the Field Equations

We now discuss the static solutions of the field equations (3)–(6) in the special case proposed by Barker with \( \omega \) in the form
\[
\omega = \frac{4 - 3\phi}{2\phi - 2}.
\]

The question of overdeterminancy in solving the field equations is settled by satisfaction of all the field equations by actual substitution of the solutions derived.

It can be easily verified that when the scalar field \( \phi \) is a constant, the field equations (3)–(6) yield a solution which describes an empty fiat space-time of Einstein’s theory.

When \( \phi \) is not a constant but is a function of \( r \) only using (7), the field equations