IS THE KOLMOGOROFF MODEL APPLICABLE TO LARGE-SCALE TURBULENCE?

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Abstract. We discuss the difficulties encountered when the Heisenberg-Kolmogoroff model for turbulence is applied to the large-scale turbulence in: (A) molecular clouds (specifically the velocity vs size relationship) and (B) stars (specifically, the estimate of convective fluxes).
A new model for large-scale turbulence is, therefore, needed.

1. Introduction

The main difficulty encountered in constructing a model of turbulence lies in the well-known 'closure problem', whereby the equation for \( \langle v^n \rangle \) (\( v \) is the fluctuating or turbulent velocity), depends on terms of the form \( \langle v^{n+1} \rangle \) which in turn satisfy an equation involving \( \langle v^{n+2} \rangle \), giving rise to an infinite chain of connected equations. For \( \langle v^2 \rangle \), one has the well-known energy equation (cf. Batchelor, 1970)

\[
\varepsilon(k) = \{ v + v_t(k) \} \int_{k_0}^{k} 2k^2 F(k) \, dk.
\] (1)

The energy \( \varepsilon(k) \) (per unit mass and time) fed into the system in the interval \( k_0 - k \) is partly dissipated by viscous forces \( \sim v(\nabla v)^2 \sim v k^2 v^2 \), and partly transferred to higher \( k \) by the nonlinear terms \( \sim v^3 \). Following the original suggestion by Heisenberg, the transfer process is written as the product of two terms. The first term represents the loss of energy by the eddies in the interval \( k_0 \) to \( k \) while the second term, represented by the action of a turbulent viscosity \( v_t(k) \), describes the redeposition of the same energy to the eddies in the remaining interval from \( k \) to \( \infty \),

\[
v_t(k) = \int_k^{\infty} v_t^{(k)} \, dk/k,
\] (2)

where \( v_t^{(k)} \) represents the eddy viscosity exerted by turbulence on a band of wavenumbers centered around \( k \). Clearly, the 'closure problem' is equivalent to prescribing the function \( v_t^{(k)} \).

In (1), \( F(k) \) is the energy spectral function, i.e., \( \frac{1}{2} F(k) \, dk \) is the energy contained in

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the wavenumber interval between $k$ and $k + dk$,

$$v^2(k) = \int_k^\infty F(k)\,dk.$$  \hfill (3)

Integrating (1) over all $k$ we obtain

$$\varepsilon = 2\nu \int_{k_o}^\infty k^2 F(k)\,dk, \quad \varepsilon \equiv \varepsilon(\infty) = \text{constant},$$  \hfill (4)

which expresses the *global* energy conservation, i.e., the nonlinear interactions transfer energy without dissipation.

A theory of turbulence aims at predicting the function $F(k)$, i.e., how turbulent energy is distributed among eddies of different sizes. It is useful to visualize a turbulent medium as a conglomerate of eddies of sizes ranging from large eddies ($\approx$ dimension of the system itself), to eddies small enough (large $k$’s) for molecular forces to operate. The dynamics of the large eddies is critically dependent on the nature of the energy feeding mechanism, i.e., on the structure of the function $\varepsilon(k)$, which may depend on magnetic fields $B$, rotation $\Omega$, etc. In general, $\varepsilon = \varepsilon(k, B, \Omega, \ldots)$. The large eddies receive the stirring energy and transfer it, via the nonlinear interactions and without dissipation, to all the eddies of smaller sizes. Finally, there are very small eddies whose dynamics depends strongly on the nature of the kinematic viscosity.

Let us now consider Equation (1) for which we need two ingredients: $v_t(k)$ and $\varepsilon(k)$. The first successful model of turbulence was worked out independently by Heisenberg and Kolmogoroff in the late forties (Batchelor, 1970). HK selected a special group of eddies sufficiently removed from the energy source to be independent of the specific nature of the stirring mechanism, but at the same time not too close to the high $k$ region where dissipative forces are most effective. The first assumption implies that in the HK region, the stirring energy has suffered many cascading processes, thereby losing memory of its specific nature. The only remaining feature is, therefore, the total energy — i.e.,

$$\varepsilon(k) = \varepsilon = \text{constant}$$  \hfill (5)

independent of $k$.

Let us now analyze $v_t(k)$. Since the HK eddies are at an intermediate distance from the two regions (low and high $k$) where forcing occurs, they can be considered ‘freely’ evolving. This means that their mean free path $\lambda_k$ can be identified with their size $l_k \sim k^{-1}$. Since, in general,

$$v_t^{(k)} \sim \lambda_k v_k$$  \hfill (6)

and

$$v^2(k) = \int_k^\infty v_k^2\,dk/k,$$  \hfill (7)