Abstract. It is shown that a set of three gyroscopes in a satellite can test vital aspects of general relativity in a period of a few days.

R. F. O'Connell discussed (O'Connell, 1969) an interesting contribution to the precession of a gyroscope due to the quadrupole moment of the earth. The stated precision of a gyroscope necessary to measure this small effect, together with the quotation of the McVittie expression for $\text{d}s^2$, proved an inspiration to draw attention to little known test results, that might be obtained in a matter of days instead of years.

1. Geometrical Viewpoint

The relevant McVittie expression for $\text{d}s^2$ is (cf. McVittie, 1956)

$$\text{d}s^2 = (1 - 2m/r') \text{d}t^2 - (1 - 2m/r')^{-1} (r'^2 \text{d}^2 + r'^2 \sin^2 \theta \text{d} \theta^2),$$  \hspace{1cm} (1)

where $m$ stands for the so-called mass radius of the Earth, being $GM/c^2 = 4.5$ mm, so that $m/r'$ has a value of about $7 \times 10^{-10}$ for a gyroscope in a circular orbit around the Earth, that is moving at a moderate altitude.

This expression has been obtained by neglecting terms of order $(m/r')^2$, so that we could have followed McVittie in writing $(1 + 2m/r')$ instead of $(1 - 2m/r')^{-1}$, but in the above given form we detect more easily the resemblance and difference with the familiar expression

$$\text{d}s^2 = (1 - 2m/r) \text{d}t^2 - (1 - 2m/r)^{-1} \text{d}r^2 - (r^2 \text{d}^2 + r^2 \sin^2 \theta \text{d} \theta^2)$$  \hspace{1cm} (2)

as given, among others, by Fokker (1929), Bergmann (1946), but also by McVittie on a previous page of his book. Although $r'$ and $r$ are not necessarily equivalent, and can be related even in such a manner as to make (1) and (2) identical, it remains true that both expressions, as they stand, purport to convey information about the geometry of space-time, so that we may ask if one or the other is a better representation of physical reality. The point is that both $r'$ and $r$ are to a certain extent arbitrary quantities, so that an identification with the astronomical value of a radial distance from a given mass centre is not necessarily meaningful, apart from the fact that the possible accuracy of direct observations falls far short of the required accuracy to decide the issue (McVittie, 1956, p. 86; Blokland, 1968, p. 492).

Resorting to indirect results, we know three classical checks on relativity, to which has been added the as yet untried possibility of observing the cumulative effect of
space-time geometry on the precession of a gyroscope, as might become measurable after long periods of time (Tonnelat, 1964).

However, if the orientation of a gyroscope does indeed behave according to the law of parallel transport of a vector in general relativity, it then can be shown that the stated accuracy of gyroscopes now allows a far more rapidly obtainable result.

Displacing a vector $A^i$ parallel to itself along a satellite orbit in the equatorial plane needs the use of only two Christoffel symbols of the second kind, because $i$ then takes the values $r$ and $\theta$ only (cf. Bergmann, 1946; p. 68).

$$dA^i = -\left\{ \begin{array}{c} i \\ k l \end{array} \right\} A^k dx^l.$$ (3)

The relevant covariant metric tensor components out of (1) are

$$g_{rr} = -(1 - 2m/r')^{-1} \quad \text{and} \quad g_{\theta\theta} = -r^2(1 - 2m/r')^{-1},$$ (4)

and those out of expression (2) are

$$g_{rr} = -(1 - 2m/r)^{-1} \quad \text{and} \quad g_{\theta\theta} = -r^2.$$ (5)

The vector increments following out of (1) are (putting $r' = r$)

$$dA^\theta = -g^{\theta\theta}[r\theta, \theta] A^r d\theta = r^{-2}(1 - 2m/r)[-(r - 3m)(1 - 2m/r)^{-2}]$$

$$\times A^r d\theta = -r^{-1}(1 - m/r) A^r d\theta,$$ (6)