BIANCHI TYPE-\( V_{I_0} \) CONFORMALLY INVARIANT
SCALAR FIELDS

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Abstract. An exact solution of Einstein’s equations corresponding to the conformally invariant scalar field with tracefree energy-momentum tensor as source is obtained in Bianchi type \( V_{I_0} \) class of metrics. The solution represents a spatially homogeneous but anisotropy universe which admits anisotropic expansions. Some properties of the cosmological model are discussed.

1. Introduction

Callen et al. (1970) proposed a new theory of gravitation in which the renormalized energy-momentum tensor defines the same four-momentum and Lorentz generators as the conventional tensor. The theory meets all the experimental tests that have been applied to general relativity. In recent years considerable interest has been shown to the study the conformally invariant scalar field as the matter field in the framework of this new theory (Frøyland, 1982; Vaidya and Som, 1983). Accioly et al. (1983) obtained solutions of the field equations to the conformally invariant scalar field with tracefree energy-momentum tensor as source in Bianchi type-I space-times. However, since Bianchi type-I models are a very special set of spatially homogeneous models, it is of considerable interest to consider the more general Bianchi type \( V_{I_0} \) space-times to study the large-scalar dynamics of the universe (Ryan and Shepley, 1975).

In this paper we obtain an exact solution of Einstein’s equations to the conformally invariant scalar field with tracefree energy-momentum tensor in Bianchi type-\( V_{I_0} \) models. The solution corresponds to an anisotropic and spatially homogeneous cosmological model. The solution for the plane symmetric Bianchi type-I model is obtained as the special case.

2. Field Equations and Solutions

The gravitational field equations in the conformally invariant scalar field theory with tracefree energy-momentum tensor (cf. Vaidya and Som, 1983) are

\[
R_{\mu \nu \lambda \sigma} f(u) = g_{\mu \nu} u_{; \lambda} u^{; \sigma} - 4 u_{; \mu} u_{; \nu} + 2 uu_{; \mu ; \nu},
\]

\[ R = 0, \]

\[ u_{; \mu ; \mu} = 0, \]

where a comma denotes partial derivative, a semicolon denotes covariant derivatives.
derivative; and
\[ f(u) = 1 - u^2, \quad (2.4) \]
\[ u = (k/6)^{1/2} \phi; \quad (2.5) \]
\( \phi \) being massless scalar field. Other symbols have their usual meaning.

The metric of this class of models is
\[ ds^2 = -dt^2 + A^2(t) dx^2 + B^2(t)e^{-2q_\phi} dy^2 + C^2(t)e^{2q_\phi} dz^2, \quad (2.6) \]
where \( A, B, C \) are cosmic scale functions, and \( q \) is a non-zero constant. We number the coordinates \( x, y, z, \) and \( t \) as 1, 2, 3, and 4, respectively.

The field equations (2.1)-(2.5) for the metric (2.6) can be written as
\[ (\frac{A_{44}}{A} + \frac{A_4 B_4}{AB} + \frac{A_4 C_4}{AC} - \frac{2q^2}{A^2}) f(u) = u_4^2 + 2uu_4 A_4, \quad (2.7) \]
\[ (\frac{B_{44}}{B} + \frac{A_4 B_4}{AB} + \frac{B_4 C_4}{BC}) f(u) = u_4^2 + 2uu_4 B_4, \quad (2.8) \]
\[ (\frac{C_{44}}{C} + \frac{A_4 C_4}{AC} + \frac{B_4 C_4}{BC}) f(u) = u_4^2 + 2uu_4 C_4, \quad (2.9) \]
\[ (\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{C_{44}}{C}) f(u) = -3u_4^2 + 2uu_{444}, \quad (2.10) \]
\[ q \left( \frac{B_4}{B} - \frac{C_4}{C} \right) f(u) = 0, \quad (2.11) \]
\[ u_{44} + u_4 \left( \frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C} \right) = 0, \quad (2.12) \]
\[ \frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{A_4 B_4}{AB} + \frac{B_4 C_4}{BC} + \frac{A_4 C_4}{AC} - \frac{q^2}{A^2} = 0; \quad (2.13) \]
where the subscript 4 denotes differentiation with respect to \( t \). Equations (2.7)-(2.12) are not independent because of the Equation (2.13). Therefore, we need not consider Equation (2.10).

Equation (2.11) readily gives
\[ B = \mu C, \quad (2.14) \]
\( \mu \) being an integration constant. Without a loss of generality, we take \( \mu = 1 \). In order to treat remaining equations we introduce the new variables \( \alpha, \beta, \) and \( T \) by
\[ A = e^\alpha, \quad B = e^\beta, \quad dt = AB^2 dT; \quad (2.15) \]
and differentiation with respect to \( T \) is denoted by a dash. Equation (2.12) then becomes
\[ u'' = 0, \quad (2.16) \]