A STATIC THEORY OF THE STABILITY OF THE EQUILIBRIUM*

GIULIO CALAMAI
Osservatorio Astrofisico di Arcetri, Firenze, Italy

Abstract. A static relativistic theory of the stability of the equilibrium of an isentropic spherically-symmetric star is deduced from the properties of a function $u$ which is solution of a second-order differential equation, and which is related to the model by means of the formula $u = \frac{\partial m(v, \varrho_c)}{\partial \varrho_c}$, where $m$ is the mass-energy inside the coordinate volume $v$ and $\varrho_c$ is the central mass-energy density.

1. Introduction

The stability of the equilibrium has been analyzed by the Author in two previous papers (Calamai, G., 1970a, b, hereafter referred to as Paper I and Paper II). In Paper I it is shown that the equilibrium of an isentropic spherically symmetric star, whose surface mass-energy density $\varrho_s$ is zero and whose total baryon number $A$ (or mass-energy $M$) is kept constant, is stable if, and only if, the solution $u(v)$ of the differential equation

$$\frac{d}{dv} \left( R \frac{du}{dv} \right) + Su = 0$$ (1.1)

with the initial condition $u(0)=0, u'(0)=1$ has a zero at the center ($v=0$) and no other zero inside the star.

Putting

$$f[v, m(v), m'(v)] = \frac{n_b(m'(v))}{\left(1 - \frac{2m(v)}{r}\right)^{1/2}},$$ (1.2)

we obtain

$$R = \frac{\partial^2 f}{\partial m'^2}; \quad S = \frac{d}{dv} \frac{\partial^2 f}{\partial m \partial m'} - \frac{\partial^2 f}{\partial m^2}.$$ (1.3)

where $r=(3v/4\pi)^{1/3}$ is the radial Schwarzschild coordinate, $m(v)$ is the mass-energy inside the sphere whose coordinate volume is $v=4\pi r^3/3$, $n_b$ is the baryon density by number, and the mass-energy density $\varrho(v)$ is equal to $m'(v)$. In Paper II a physical meaning is ascribed to the function $u$; for a sequence of model labelled by the central density $\varrho_c$ the formula holds

$$u(v, \varrho_c) = \frac{\partial m(v, \varrho_c)}{\partial \varrho_c};$$ (1.4)

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and, hence, at the surface of the star, where \( v = V = 4\pi R^3/3 \), it is

\[
\frac{dM}{dq_c} = u(V, q_c),
\]

(1.5)
a formula which determines the dependence of the mass-energy \( M \) on the central mass-energy density \( q_c \).

The purpose of the present paper is to work out a complete and selfconsistent static theory of the equilibrium stability based on the properties of the function \( u(v, q_c) \). We shall also try to get some information on the source of instability and the mechanism by which instability is set up.

2. The Stability of the Equilibrium of Cold Stars

Let us discuss first the relatively simple case of the equilibrium configuration of cold matter catalyzed to the endpoint of thermonuclear evolution. Since the cold-catalyzed matter obeys an unique and universal equation of state (Harrison et al., 1958; and Wheeler, 1962, 1964), the solutions of the problem are functions of the only parameter \( q_c \), given by

\[
m = m(v, q_c);
\]

(2.1)
and the mass-energy \( M \) and the radius \( R \) are determined by \( q_c \) from

\[
M = M(q_c); \quad R = R(q_c).
\]

(2.2)
These functions can be inverted to give

\[
q_c = q_c(R) \quad \text{and} \quad M = M(R).
\]

(2.3)
Since the following discussion is based on the function \( u \) which is a solution of the Equation (1.1) it is worthwhile to recall the properties of the solutions of a linear homogeneous differential equation of the second order. The coefficients of (1.1) are regular functions in the range considered and hence \( u(v) \) is also regular; two of the three functions \( u, u', u'' \) cannot be zero for the same value of \( v \) (except of course for the trivial case \( u(v) \equiv 0 \)); the zeros of \( u(v) \) are then simple and \( u \) changes its sign crossing a zero; between two consecutive zeros (conjugate points) of \( u(v) \) there must be an odd number of zeros of \( u'(v) \); two solutions, whose ratio is a constant, have the same zeros.

The function \( M(R) \) is not single-valued, since in the diagram \( M \) vs \( R \) there are several (easy distinguishable) points where \( dM/dR \) becomes infinite. At other points (critical points) \( dM/dR = 0 \). All these points are of fundamental importance for the analysis of the stability of the equilibrium and, hence, following our objective of basing the theory on the function \( u(v, q_c) \), we shall relate the existence of these points to the properties of this function.