AN EXACT SOLUTION OF TWO-DIMENSIONAL STEADY MGD FLOWS

C. THAKUR and R. B. MISHRA

Department of Mathematics, Faculty of Science, Banaras Hindu University, Varanasi, India

(Received 7 June, 1989)

Abstract. Steady-plane flow of an inviscid, electrically-conducting, compressible fluid with infinite electrical conductivity is considered and a single partial differential equation is obtained which involves two functions. Appropriate specialization of these functions generate new exact solutions of the original equations.

1. Introduction

In recent years the study of flow problems of electrically-conducting fluids has received considerable interest. Such studies have been made for many years in connection with astrophysical and geophysical problems such as sunspot theory, motion of the interstellar gas, origin of Earth magnetism, etc. Recently, some engineering problems such as controlled fusion research, re-entry problems of intercontinental ballistic missiles, plasma jet, communication power conversion need the studies of the flow of an electrically-conducting fluid.

In general, exact solutions in magnetogasdynamics are rare. Nemenyi and Prim (1948) and Prim (1952) studied the geometry of a steady, plane, rotational, non-MHD flow of perfect gas and gases with product equation of state when the velocity magnitude is constant on each individual streamline. Martin (1950) and Chandna and Smith (1971) also studied this problem and generalized the results for any gas. By use of the same method Chandna and Nath (1973) determine the geometry of orthogonal plane magnetogas-dynamic flows when the velocity magnitude is constant on each individual streamline. Chandna and Barron (1981) have studied constantly inclined MGD flows by using Martin's (1971) approach. Also Chandna et al. (1985) have studied the steady-plane MGD flow applying Monges, Charpits, and the hodograph methods.

In this paper a new approach is applied for steady-adiabatic flow of an inviscid, electrically-conducting, compressible fluid. A single partial differential equation which involves two functions, is obtained. Appropriate specialization of these functions generate new exact solutions of the original equation.

2. Basic Equations

The steady adiabatic flow of an inviscid, electrically-conducting, compressible fluid of infinite electrical conductivity is governed (cf. Pai, 1962) by

$$\text{div}(\rho \mathbf{v}) = 0,$$  \hspace{1cm} (2.1)
\[ \rho (\mathbf{v} \cdot \text{grad}) \mathbf{v} + \text{grad} p = (\text{curl} \mathbf{B}) \times \mathbf{B}, \quad (2.2) \]
\[ \text{curl}(\mathbf{v} \times \mathbf{B}) = 0, \quad (2.3) \]
\[ \text{div} \mathbf{B} = 0, \quad (2.4) \]
\[ \mathbf{v} \cdot \text{grad} s = 0, \quad (2.5) \]

where \( \mathbf{v} \) is the velocity vector, \( \mathbf{B} \) the magnetic field vector, \( \rho \) the gas density, \( s \) the specific entropy, and \( p \) the pressure. By use of the identity
\[ (\nabla \times \mathbf{F}) \times \mathbf{F} = (\mathbf{F} \cdot \nabla) \mathbf{F} - \frac{1}{2} \text{grad} F^2, \]
Equation (2.2) can be written as
\[ \rho (\mathbf{v} \cdot \text{grad}) \mathbf{v} + \nabla p + \frac{1}{2} \nabla \mathbf{B}^2 = (\mathbf{B} \cdot \text{grad}) \mathbf{B}, \quad (2.6) \]

where \( B \) is the magnitude of magnetic field \( \mathbf{B} \). We now consider the flow to be two-dimensional so that \( \mathbf{v} \) and \( \mathbf{H} \) lie in a plane defined by the rectangular coordinates \( x, y \) and all the flow variables are functions of \( x, y \). Therefore, the above system of equations is replaced by the system
\[ \frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) = 0, \quad (2.7) \]
\[ \frac{\partial p^*}{\partial x} + \rho \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = B_1 \frac{\partial B_1}{\partial x} + B_2 \frac{\partial B_1}{\partial y}, \quad (2.8) \]
\[ \frac{\partial p^*}{\partial y} + \rho \left( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = B_1 \frac{\partial B_2}{\partial x} + B_2 \frac{\partial B_2}{\partial y}, \quad (2.9) \]
\[ u B_2 - v B_1 = h, \quad (2.10) \]
\[ \frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} = 0, \quad (2.11) \]
\[ u \frac{\partial s}{\partial x} + v \frac{\partial s}{\partial y} = 0. \quad (2.12) \]

This is a system of seven equations where in \( (u, v) \) are the velocity components \( B_1, B_2 \) are the components of the magnetic field vector, \( p^* = p + \frac{1}{2} \mathbf{B}^2 \) and \( h \) is an arbitrary constant which is zero for aligned flows and non-zero in the case of non-aligned flows. Employing Equations (2.7) and (2.11), Equations (2.8) and (2.9) are replaced by
\[ \frac{\partial}{\partial x} (p^* + \rho u^2 - B_1^2) + \frac{\partial}{\partial y} (\rho uv - B_1 B_2) = 0, \quad (2.13) \]