RAPIDLY ROTATING POLYTROPES IN THE POST-NEWTONIAN APPROXIMATION TO GENERAL RELATIVITY

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Abstract. The study of uniformly polytropes with axial symmetry is extended to include all rotational terms of order $\Omega^4$, where $\Omega$ is the angular velocity, consistently within the first post-Newtonian approximation to general relativity. The equilibrium structure is determined by treating the effects of rotation and post-Newtonian gravitation as independent perturbations on the classical polytropic structure. The perturbation effects are characterized by a rotation parameter $\epsilon = \Omega^2/2\pi G Q_0$, and a relativity parameter, $\sigma = p_0/\rho_0 c^2$, where $p_0$ and $\rho_0$ are the central pressure and density respectively. The solution to the structural problem is obtained by following Chandrasekhar’s series expansion technique and is complete to the post-Newtonian rotation terms of order $\sigma^3$. The critical rotation parameter $v_\text{c}$, which characterizes the configuration with maximum uniform rotation, is accurately evaluated as a function of $\sigma$. Numerical values for all the structural parameters needed to determine the equilibrium configurations are presented for polytropes with indices $n = 1, 1.5, 2, 2.5, 3$, and $3.5$.

1. Introduction

Uniformly rotating polytropes were first studied by Chandrasekhar (1933) with a perturbation technique in which all terms of order $\Omega^2$, where $\Omega$ is the angular velocity, were included in the equations. The resulting theory was strictly applicable only to the case of slow rotation. In more recent times, Roberts (1963a, b), James (1964) and Hurley and Roberts (1964) have applied modern computational methods to solve the exact equations of the problem. These methods, while inherently more accurate than a perturbation method, are also more laborious. This led Monaghan and Roxburgh (1965) to develop an improved first order perturbation method for solving the problem of rapidly rotating polytropes. A different approach was taken by Anand (1968) and Ochionero (1968) who extended Chandrasekhar's analysis to include the terms of order $\Omega^4$ in the theory. This extension of the first order theory, however, was regarded primarily as a step forward including the effects of post-Newtonian gravitation into the problem. Apart from the intrinsic interest of rotating relativistic configurations, Fowler (1964, 1966) and Roxburgh (1965) had shown that rotation could overcome the relativistic instability occurring in supermassive stars. For this reason alone, it is of interest to investigate rapidly rotating configurations in the post-Newtonian approximation to general relativity. The relevant equations of hydrodynamics in which all terms of order $1/c^2$ are consistently included have been developed by Chandrasekhar (1965a). Kretetz (1967a) later applied these equations to slowly rotating configurations and determined the first order relativistic correction to Clairaut’s equation.

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More rapidly rotating configurations have recently been studied by Chau (1969) with the method of Hurley and Roberts.

Our purpose here is to present a consistent theory of rotating polytropes which includes all second order rotational terms and independently, all effects of post-Newtonian gravitation. We shall use a slightly modified version of the Monaghan-Roxburgh method to improve our accuracy in the case of very rapid rotation. We assume uniform rotation in the sense that a distant observer sees the configuration rotating as a rigid body (see Thorne, 1967). The theory is developed in the particular co-ordinate system of Chandrasekhar (1965a) and for uniform rotation and axisymmetry about the $x_3$-axis, the non-zero metric components are

$$
g_{00} = 1 - 2 \frac{U}{c^2} + \frac{1}{c^4} (2U^2 - 4\Phi),$$

$$
g_{01} = -\frac{1}{c^3} 4 \Omega x_2 \phi, \quad g_{02} = \frac{1}{c^3} 4 \Omega x_1 \phi,$$

$$
g_{11} = g_{22} = g_{33} = - \left(1 - \frac{2U}{c^2}\right).$$

The potentials $U$, $\Phi$ and $D$ completely determine the basic physical properties of the rotating configuration and thus, the problem reduces to finding solutions for these functions. The basic equations of the problem are set out in Section 2 and in Section 3 we obtain approximate solutions for the potentials $U$, $D$ and $\Phi$ that are complete to rotation terms of order $\Omega^4$. In Section 4 we discuss the boundary of the configuration and obtain the parameters characterizing the configuration with maximum uniform rotation. A brief discussion of the results is given in the final section.

2. The Structure Equations for Polytropes

The post-Newtonian equations of hydrostatic equilibrium for an ideal fluid that is uniformly rotating and has axial symmetry about the $x_3$-axis is (Krefetz, 1967a)

$$
\left[1 - \frac{1}{c^2} (\Pi + p/q)\right] \nabla p = q \nabla \mathcal{U},
$$

where

$$
\mathcal{U} = U + \frac{1}{2} \Omega^2 \sigma^2 + \frac{1}{c^2} \left[\frac{1}{4} (\Omega^2 \sigma^2)^2 + 4U \left(\frac{1}{2} \Omega^2 \sigma^2\right) + 2\Phi - 4\Omega^2 \sigma^2 \phi\right],
$$

and $q$ is the mass density, $p$ the pressure, $q\Pi$ the internal energy density, $\Omega$ the angular velocity, $\phi$ the distance from the rotation axis. The source equations for the potentials $U$, $\Phi$ and $\phi$ are

$$
\nabla^2 U = -4\pi Gq,
$$

$$
\nabla^2 \phi + \frac{2}{\sigma} \frac{\partial \phi}{\partial \sigma} = -4\pi Gq,
$$

and

$$
\nabla^2 \Phi = -4\pi Gq \left[\Omega^2 \sigma^2 + U + \frac{1}{2} \Pi + \frac{1}{3} p/q\right].
$$