Space-Time Groups for the Lattice

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In the assumption of a lattice theory in which the continuous limit is not taken, the metric of the discrete space-time should be invariant under integral transformations. Based on local isomorphisms between real forms, a method is proposed in order to find the rational and integral elements of the pseudoorthogonal groups. Besides, the rational and integral trigonometric and hyperbolic functions are constructed on the lattice.

1. ISOMORPHISM BETWEEN REAL FORMS

According to Cartan theory, there are some real forms of simple Lie groups of low dimensionality which are locally isomorphic (Helgason, 1978). We describe them by the bijection of $\mathbb{R}^n$ onto a set of matrices $A$.

(i) $SL(2, \mathbb{R}) \approx SO(2, 1)$. Define a set of $2 \times 2$ real matrices $A$, by the conditions $A^T = A$, where $A^T$ means transposed. The bijection of an element $(x_0, x_1, x_2)$ of $\mathbb{R}^3$ onto a matrix $A$ is the following:

$$A = \begin{pmatrix} x_0 + x_2 & x_1 \\ x_1 & x_0 - x_2 \end{pmatrix}$$

The transformations $A' = SAS^T$, with $S \in SL(2, \mathbb{R})$, map $A$ into itself. Since

$$\det A = x_0^2 - x_1^2 - x_2^2 = \det A'$$

this transformation induces the desired isomorphism.

(ii) $SL(2, \mathbb{C}) \approx SO(3, 1)$. Define $A$, a $2 \times 2$ complex matrix, by the condition $A^\dagger = A$, where $A^\dagger$ means the Hermitian conjugate matrix. The bijection of $(x_0, x_1, x_2, x_3)$ in $\mathbb{R}^4$ onto $A$ is given by

$$A = \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix}$$

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The transformation $A' = SAS^+$ with $S \in SL(2, \mathbb{C})$ maps $A$ into itself, as it is well known (Gel'fand et al., 1963). Since
\[ \det A = x_0^2 - x_1^2 - x_2^2 - x_3^2 = \det A' \]
this transformation induces the mentioned isomorphism.

(iii) $Sp(4, \mathbb{R}) \approx SO(3, 2)$. The matrix $A$ is a four-dimensional real matrix, satisfying $A^T J = JA$ and $\text{Tr} A = 0$, where
\[ J = \begin{pmatrix} 0 & \ldots & 0 \\ \ldots & \ddots & \ldots \\ 0 & \ldots & 0 \\ -1 & \ldots & 0 \end{pmatrix} \]
$1$ is the unit matrix of dimension $2$.

The bijection of an element $(x_1, x_2, x_3, x_4, x_5)$ of $\mathbb{R}^5$ onto $A$ is given by
\[ A = \begin{pmatrix} x_1 & x_2 + x_3 & 0 & x_4 + x_5 \\ x_2 - x_3 & -x_1 & -x_4 - x_5 & 0 \\ 0 & x_4 - x_5 & x_1 & x_2 - x_3 \\ -x_4 + x_5 & x_2 + x_3 & -1 \end{pmatrix} \]
The transformation $A' = SAS^{-1}$ with $S \in Sp(4, \mathbb{R})$ maps $A$ into itself, namely, $A'^T J = JA'$, $\text{Tr} A' = 0$. Since
\[ \det A = (x_1^2 + x_2^2 - x_3^2 - x_4^2 + x_5^2)^2 = \det A' \]
this transformation induces the desired isomorphism.

(iv) $Sp(1, 1) \approx SO(4, 1)$. $A$ is defined by the four-dimensional complex matrix satisfying $A^T J = JA$, $A^* K = KA$, $\text{Tr} A = 0$, with $K = \text{diag}(1, -1, 1, -1)$.

The bijection of an element $(x_1, x_2, x_3, x_4, x_5)$ of $\mathbb{R}^5$ onto $A$ is
\[ A = \begin{pmatrix} x_1 & x_2 + i x_3 & 0 & x_4 + i x_5 \\ -x_2 + i x_3 & -x_1 & -x_4 - i x_5 & 0 \\ 0 & x_4 - i x_5 & x_1 & -x_2 + i x_3 \\ -x_4 + i x_5 & 0 & x_2 + i x_3 & -x_1 \end{pmatrix} \]
Given an element $S$ of the group $Sp(1, 1)$, that is to say, $S^T JS = J$, $S^* K S = K$, the transformation $A' = SAS^{-1}$ maps $A$ into itself. Since
\[ \det A = (x_1^2 - x_2^2 - x_3^2 - x_4^2 - x_5^2)^2 = \det A' \]
this transformation induces the desired isomorphism.

(v) $SU(2, 2) \approx SO(4, 2)$. $A$ is defined by the four-dimensional complex matrix, satisfying $A^T = -A$, $A^* I = I A$, with $\bar{A}$, the complex conjugate matrix of $A$, $A^*$ the dual matrix of $A$, namely, $(A^*)_{ab} = \frac{1}{2} \in_{abcd} A^{cd}$, and
\[ I = \begin{pmatrix} 1 & \ldots & 0 \\ \ldots & \ddots & \ldots \\ 0 & \ldots & 1 \end{pmatrix} \]