Comments on Limits of Space-Times

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A coordinate-dependent investigation of limits of space-times is given. Different kinds of limits are found and analyzed. A criterion selecting limits which appear to be nothing but "technical tricks" is proposed. Finally, a method for generating new limits from a given family of metrics is examined.

1. INTRODUCTION

In the study of general relativity, it frequently occurs that solutions of Einstein's field equations may be collected into families of space-times depending on some free parameters. This enables one to consider limits of space-times when a given parameter \( \lambda \) approaches a certain fixed value \( \lambda_0 \) (e.g., \( \lambda_0 = 0 \)). In this context, one usually understands limits in a strictly coordinate-dependent manner. Owing to this fact, one of the most surprising results is, perhaps, the possibility of finding different space-times as limit.

A precise formulation of this problem was given by Geroch (Geroch, 1969). There, the possibility of finding new solutions of Einstein's equations as limit of known solutions was pointed out. This is in close connection with the formidable problem of determining all the limits of an assigned metric or, otherwise, of deciding whether a given class of solutions is closed or not. Insights may be obtained by understanding how the limit operation works.

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2A family of solutions of Einstein's equations is called a closed class if it contains all its limits (Geroch, 1969).
The main purpose of this note is to describe, by means of a critical analysis of the actual algorithm, how different space-times may be obtained as limits of a given family of metrics as $\lambda \to 0$. In this context, we enunciate a property in order to characterize limits which are nothing but "technical tricks"; as a further result, the present investigation provides a method for generating limits from a given family of metrics.

2. ANALYSIS OF THE LIMIT PROCEDURE

In the following it is useful to adopt a coordinate-dependent approach as this lies special stress on the possibility of finding different space-times as $\lambda \to 0$. Although this approach involves only local properties, no generality is lost in view of the theorem that "the global (maximal) limit of a family of space-times is uniquely determined by the local knowledge of the limit" (Geroch, 1969).

Accordingly, consider the family of vacuum metrics

$$\begin{align*}
 ds^2 &= -\left(\lambda^2 - \frac{\beta^2}{r}\right) dt^2 + \left(\lambda^2 - \frac{\beta^2}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \lambda^{-2}\sin^2 \lambda \theta \, d\phi^2) \\
     &= \frac{\lambda}{X} 
\end{align*}$$

(2.1)

In particular, putting $\lambda = 1$, $\beta^2 = 2m$, we obtain the Schwarzschild vacuum metric. On the other hand, the choice $\lambda = 0$, $\beta^2 = 2$ yields the Kasner metric (Kasner, 1921). Now, assuming $\lambda \neq 0$, the following coordinate transformation,

$$
 r = \lambda \rho, \quad t = \tau / \lambda, \quad \theta = \xi / \lambda
$$

(2.2)

makes the metric (2.1) into the form

$$
 ds^2 = \left(1 - \frac{\beta^2}{\lambda^3 \rho}\right) d\tau^2 + \left(1 - \frac{\beta^2}{\lambda^3 \rho}\right)^{-1} d\rho^2 + \rho^2(d\xi^2 + \sin^2 \xi \, d\phi^2)
$$

(2.3)

Once again, we obtain the usual form of the Schwarzschild metric by putting $\beta^2 / \lambda^3 = 2m$.

The metric (2.3) has no limit as $\lambda \to 0$. Nonetheless, it is possible to restore the parameter $\lambda$ into its original position by means of the inverse of the transformation (2.2) thus obtaining the Kasner metric as $\lambda \to 0$. We remark this is exactly Geroch's example. Therefore, in order to obtain the required interpretation of the limit process, it is necessary to analyze the role of the preliminary coordinate transformation.