We generalize the first and second Noether theorems (Noether identities) to a constrained system in phase space. As an example, the conservation law deriving from Lagrange's formalism cannot be obtained from $H_E$ via the generalized first Noether theorem (GFNT); Dirac's conjecture regarding secondary first-class constraints (SFCC) is invalid in this example. A preliminary application of the generalized Noether identities (GNI) to nonrelativistic charged particles in an electromagnetic field shows that on the constrained hypersurface in phase space one obtains electric charge conservation. This conservation law is valid whether Dirac's conjecture holds true or not.

1. INTRODUCTION

The connection between the invariance of the action integral under a finite continuous group and conservation laws is given by the first Noether theorem (FNT). The second Noether theorem refers to the invariance of the action under an infinite continuous group. In this case there exist differential identities which involve variational derivatives. These theorems have an important role in theoretical physics. A generalization of the FNT was given by Rosen (1974a, b and references therein), and a generalization of the FNT for constrained and nonconservative systems was given by Li (1981, 1984; Li and Li, 1990). A generalization of Noether's identities for variant systems was given by Li (1987). In these papers, all considerations are based on an examination of the Lagrangian in configuration space and the corresponding transformation expressed in terms of Lagrange variables. For regular Lagrangians of classical mechanical systems the invariance of the Lagrangian under a finite continuous group in terms of Hamilton’s variables was discussed by Djukic (1974). Here, we further discuss singular Lagrangian systems. Dirac (1950, 1964) proposed a method for developing
the formalism for this system, and its quantization. The singular Lagrangian system is subject to some inherent phase space constraint. Dirac conjectured that all secondary first-class constraints (SFCC) generate gauge transformations which leave the physical state invariant. Dirac's conjecture has been widely discussed. In this paper, first, we generalize the FNT in phase space for a constrained Hamiltonian system. An example is given in which the conservation law deriving from the usual Lagrange formalism cannot be obtained from $H_E$ via this GFNT, which implies that Dirac's conjecture fails in this example. Second, considering the transformation properties of the system under an infinite continuous group in terms of canonical variables, we obtain the GNI in phase space. Combining these GNI and constraint conditions, we obtain more relations among some of the variables. A preliminary application of the GNI to nonrelativistic charged particles in an electromagnetic field, on the constrained hypersurface, shows that one obtains electric charge conservation automatically, which differs from the usual way to obtain this result. This conservation law is valid whether Dirac's conjecture holds or not.

2. GENERALIZATION OF FIRST NOETHER THEOREM

Consider the transformation properties of a constrained dynamic system under a finite continuous group. We can generalize the FNT to Hamiltonian coordinates. For simplicity one usually considers a system with finite degrees of freedom exhibiting the essential problems of invariant theories; the extension to field theories is formally straightforward. Consider a mechanical system with $N$ degrees of freedom described by a singular Lagrangian $L(t, q, \dot{q})$ ($q = \{q^1, \ldots, q^N\}$). This system is subject to Dirac's constraints

$$G_k(q, p) = 0 \quad (k = 1, 2, \ldots, K)$$

where $p = \{p_1, \ldots, p_N\}$ are the generalized momenta corresponding to the generalized coordinates $q$. Let us consider an infinitesimal continuous $r$-parameter transformation of the time, generalized coordinates, and generalized momenta

$$t \to t' = t + \delta t = t + \varepsilon_\sigma \tau^\sigma(t, q, p)$$

$$q^i(t) \to q^i(t') = q^i(t) + \delta q^i(t) = q^i(t) + \varepsilon_\sigma \xi^\sigma_i(t, q, p)$$

$$p_i(t) \to p_i(t') = p_i(t) + \delta p_i(t) = p_i(t) + \varepsilon_\sigma \eta^\sigma_i(t, q, p)$$

Suppose $L_{PS} = p_i \dot{q}^i - H$ is gauge variant under the transformation (2), i.e., is invariant up to an exact differential $\varepsilon_\sigma \frac{d\Omega^\sigma}{dt}$, where $H$ is the Hamiltonian, $\Omega^\sigma = \Omega^\sigma(t, q, p)$ ($\sigma = 1, 2, \ldots, r$), and $\varepsilon_\sigma$ are parameters. Repeated