Representation of Complete Ortholattices
by Sets with Orthogonality

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We consider an algebraic closure operator induced by orthogonality on an arbitrary set and we investigate some problems with regard to the orthomodular law for a complete lattice of closed subsets.

The basic idea of our approach is the arrangement of a representation of a complete lattice by an algebraic closure operator, more precisely by a lattice of closed subsets. There is a modification (Birkhoff, 1967) for complete ortholattices based on a closure operator induced by the orthogonality relation. The formal definition is:

Let V be an arbitrary nonempty set with a relation of orthogonality (we will use designation \( \perp \)) such that we require symmetricity and irreflexivity (i.e., \( x \perp y \iff y \perp x \) and \( \forall x \in V, x \perp x \) is not true).

A closure operator on the system of all subsets of V is defined \( \text{CL}(A) = A_{\perp} \) such that \( A_{\perp} = \{z \in V; \forall x \in A, x \perp z\} \). It is easy to see that \( \text{CL} \) is closure operator: \( A \subseteq \text{CL}(A) \) and \( \text{CL}^2(A) = \text{CL}(A) \) and \( A \subseteq B \Rightarrow \text{CL}(A) \subseteq \text{CL}(B) \).

The reader can find in Birkhoff (1967) that the system of all closed subsets (the sets which are equal to its closure) forms a complete ortholattice. The problem of representation of an arbitrary complete ortholattice is investigated in McLaren (1964). In our approach we describe another representation such that our representation is "maximal" in some sense. Let us use the language of graph theory.

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The complete graph is a graph such that every couple \( \{x, y\} \) is in the edge set.

Any orthogonality relation can be represented by a graph with vertex set the basic set of orthogonality and \( x \) is orthogonal to \( y \) is represented by the edge \( \{x, y\} \).

The relation of complementarity is defined: \((G_1, E_1)\) and \((G_2, E_2)\) are complementary graphs if \( G_1 = G_2 \) is the relation between vertex sets and sets of edges \( E_1 \) and \( E_2 \) form a decomposition of the set of edges of a complete graph with vertex set \( G \). An induced subgraph is a restriction of graph \((G, E)\) to \((H, F)\) such that \( H \subseteq G \) and \( F = E/H \). The relation between graphs \((G, E) \prec (H, F)\) if there is an isomorphic copy of \((G, E)\) as an induced subgraph of \((H, F)\) is a pseudoorder relation (i.e., antisymmetry is not required). We will use this relation on finite graphs and in case of finite graphs antisymmetry holds if we do not distinguish isomorphic graphs.

We start from an arbitrary complete ortholattice \((L, \leq, ', 0, 1)\). We define the orthogonality relation on \( L - \{0, 1\} \):

\[
v \perp u \iff u \leq v'
\]

It is known (Zapatrin, 1990) that the lattice of closed subsets induced by this orthogonality relation is an isomorphic copy of the starting lattice.

If we start from an arbitrary orthogonality relation to a lattice of closed subsets and apply our procedure of creation of a "new" orthogonality relation on this lattice, we obtain an orthogonality relation such that the starting relation is an induced subgraph of the terminating relation. The embedding map is: \( x \rightarrow \{x\}^{1,1} \).

This is an exact formulation of the fact that there is a "maximal" orthogonality relation in a system of all orthogonality relations with the same lattice of closed subsets.

There are representations of an arbitrary complete ortholattice such that every element \( a \) of lattice is a joint of a set of join-irreducible elements which are smaller than \( a \) (McLaren, 1964; Zapatrin, 1990, n.d.). If we assume the existence of this system of join-irreducible elements for every element, we obtain a "minimal" orthogonality relation in the sense of induced subgraphs.

If we try to analyze the system of all orthogonality relations belonging to some complete ortholattice, the question of isolated points of the orthogonality relation can be posed. There is only one closed set in a lattice of closed subsets with the property "an isolated point is element of this set." It is the maximal set of the lattice of closed subsets. This implies that we can remove isolated points without influence on the terminating lattice.