Example of Order and Disorder:

$x_{n+1} = (Ax_n + B) \mod C$

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A simple model with discrete dynamics is studied. The behavior of this model is very sensitive to the particular choice of parameters determining the system.

1. In this paper I present a simple model exhibiting very complicated dynamics, which depends very strongly on the particular values of the driving parameters. For some parameters the motion can be ergodic and for others the system can reveal very ordered behavior.

In recent years there has been great interest in deterministic systems displaying irregular dynamics [see, e.g., Lichtenberg and Lieberman (1983), where many examples of such systems are given]. In contrast to the most popular systems, for the description of the proposed model only natural numbers are needed.

Section 2 contains a description of the model. Section 3 is devoted to a chaotic motion. Section 4 discussess the regular behavior of the model. Section 5 contains a brief discussion of the intermediate behavior of the model and some remarks.

2. The model is two-dimensional and the "phase space" is discrete: both time and coordinate take natural values. The set of possible positions of the material point (a particle or a "small ball") performing the motion consists of nodes regularly distributed along the circle; see Fig. 1. Let the total number of points on the circle be $C$; denote them by $0, 1, \ldots, C - 1$. The position of the ball at the instant of time $n$ will be denoted by $x_n$, so $x_n \in \{0, 1, \ldots, C - 1\}$. The "dynamics" is given by the following rule: Let

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the moving ball be at the site \( x_n \). During an elementary interval of time a ball shifts by \( Ax_n + B \) nodes in, e.g., the clockwise direction. Here \( A \) and \( B \) are natural numbers smaller than \( C < A < C - 1 \), \( 0 < B < C - 1 \). This rule can be written as the following equation of motion:

\[
x_{n+1} = (Ax_n + B) \mod C
\]

The particle constrained to the motion described by (1) is “jumping” from the node \( x_n \) at the time \( n \) to the node \( x_{n+1} \) at the time \( n + 1 \). The number \( q_n \) of whole laps of the circle is given by

\[
q_n = \left\lfloor (Ax_n + B)/C \right\rfloor
\]

where the square brackets denote the \textit{Entier} function: \( \lfloor x \rfloor \) is the greatest integer \( \leq x \). It turns out that there is a great variety of possible behaviors of the particle, according to the particular values of driving parameters \( A \), \( B \), and \( C \). On one hand, the particle can perform a random walk around the circle. On the other hand, it is possible that the particle will fall after at most two jumps into such a node in which it will remain forever, \textit{regardless of the initial value} \( x_0 \).

3. The behavior of the particle is fully determined by the properties of the sequence \( \{x_n\}^\infty_{n=0} \) generated by the iterations (1). First, note that the elements of the sequence \( \{x_n\}^\infty_{n=0} \) depend on the value \( x_0 \) and are natural numbers from the set \( \{1, \ldots, C-1\} \). Due to this, the sequence \( \{x_n\}^\infty_{n=0} \) starting at some index must be periodic, so there exist numbers \( N \) and \( T \) such that

\[
x_{n+1} = x_{n+T} \quad \forall n > N
\]

It is well known that it is possible to choose parameters \( A \), \( B \), and \( C \) such