Uncertainties and an Interpretation of Nonrelativistic Quantum Theory

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We propose an interpretation of nonrelativistic quantum theory which can be considered a generalized Copenhagen interpretation. The uncertainties (i.e., $\Delta q$ and $\Delta p$) in Heisenberg's uncertainty relation $\Delta q \cdot \Delta p \geq \hbar/2$ can be characterized as (average) errors in an approximate simultaneous measurement if the interpretation proposed here is accepted in nonrelativistic quantum mechanics. Under this interpretation, the (discrete) trajectory of a particle (like "Wilson chamber") is significant enough. We propose to analyze this trajectory numerically.

1. INTRODUCTION

Recently, we discussed Heisenberg's uncertainty relation $\Delta q \cdot \Delta p \geq \hbar/2$ (Ishikawa, 1991), gave a mathematical definition of $\Delta q$ and $\Delta p$, and proved a certain inequality which could be considered as a mathematical representation of Heisenberg's uncertainty relation. We mention some of the results obtained in Ishikawa (1991) in order to exhibit our motivations in this note.

Let $H$ be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle_H$. Let $A_0, A_1, \ldots, A_{N-1}$ be any physical quantities (i.e., self-adjoint operators) in a Hilbert space $H$. A quarter $M = (K, v, (X, \mathcal{F}, F), f = (f_0, \ldots, f_{N-1}))$ is called an approximate simultaneous measurements of $\{A_k\}_{k=0}^{N-1}$ in $H$ if it satisfies the following conditions:

1. $v$ is an element in a Hilbert space $K$ such that $\|v\|_K = 1$, and $(X, \mathcal{F}, F)$ is a projection-valued probability space in a tensor Hilbert space $H \otimes K$ and $f: X \to \mathbb{R}^N$ is a measurable map.

2. Put $\hat{A}_k = \int_X f_k(x) F(dx)(k = 0, 1, \ldots, N-1)$; then, for each $k$, a set $D_v(\hat{A}_k) (\equiv \{u \in H: u \otimes v \in D(\hat{A}_k), \text{the domain of } \hat{A}_k\})$ is a core of $A_k$, i.e., $A_k$ is essentially self-adjoint on $D_v(\hat{A}_k)$.

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3. For each $k$, $\langle u, A_k u \rangle_H = \langle u \otimes v, \hat{A}_k (u \otimes v) \rangle_{H \otimes K}$ ($u \in D_v(\hat{A}_k)$).

Furthermore, we assume that $M$ satisfies the following additional conditions:

4. For each $k$, $(\hat{A}_k - A_k \otimes I)$ on $D(\hat{A}_k) \cap D(A_k \otimes I)$ has the unique self-adjoint extension $[\hat{A}_k - A_k \otimes I]$.

5. A set $\{u \in H : u \otimes v \in \bigcap_{k=0}^{N-1} D(\hat{A}_k)\}$ is a dense set in a Banach space $H_0 (= \{u \in H : u \otimes v \in \bigcap_{k=0}^{N-1} D([\hat{A}_k - A_k \otimes I])\})$ with the norm

$$\| u \|_{H_0} = \| u \|_H + \sum_{k=0}^{N-1} \| [\hat{A}_k - A_k \otimes I](u \otimes v) \|_{H \otimes K}$$

Note that the existence of an approximate simultaneous measurement of $\{A_k\}_{k=0}^{N-1}$ satisfying conditions 4 and 5 is proved in Abu-Zeid (1987) [or Ishikawa (1991) in detail].

Now the infitness [in Ishikawa (1991) we did not dare to call it "error" or "uncertainty" since its physical meaning seemed to be not clear] $\{\Delta_M(A_k, u) : k = 0, \ldots, N-1\}$ of an approximate simultaneous measurement $M$ for $\{A_k\}_{k=0}^{N-1}$ on a state $u$ ($\| u \|_H = 1$) is defined by $\Delta_M(A_k, u) = \| [\hat{A}_k - A_k \otimes I](u \otimes v) \|_{H \otimes K}$ if $u \otimes v \in D([\hat{A}_k - A_k \otimes I]) = \infty$ otherwise. We obtained the following theorem in Ishikawa (1991).

**Theorem 1.** Let $A_0$ and $A_1$ be a pair of conjugate observables in a Hilbert space $H$ (i.e., symbolically, $A_0 A_1 - A_1 A_0 = i\hbar$). Let $M = (K, \alpha, (X, \mathcal{F}, F), f = (f_0, f_1))$ be any approximate simultaneous measurement of $A_0$ and $A_1$ satisfying the additional conditions 4 and 5. Then, the following inequality holds:

$$\Delta_M(A_0, u) \cdot \Delta_M(A_1, u) \geq \hbar/2$$

for all $u \in H (\| u \|_H = 1)$, where the left-hand side of (1) is defined as $= \infty$ if $\Delta_M(A_k, u) = \infty$ for some $k = 0, 1$.

Special and simple cases of this theorem were also investigated in Ali and Emch (1974), Ali and Prugovecki (1976), and Busch (1985).

If we take a standpoint within the so-called "Copenhagen interpretation," we believe that inequality (1) is just Heisenberg's uncertainty relation (though there is a possibility to improve the conditions 4 and 5. And we believe that the relation between the EPR argument (Einstein et al., 1935; Selleri, 1990) and Heisenberg's uncertainty relation becomes clear in Ishikawa (1991). However, the observations in Ishikawa (1991) are not necessarily physical, but rather mathematical. Hence we think that the physical meaning of the unfitness $\{\Delta_M(A_k, u) : k = 0, 1\}$ is still not clear. Furthermore, we think that it is necessary to discuss the matter from various viewpoints in order to conclude that the inequality (1) is a mathematical representation of Heisenberg's uncertainty relation.