A Note on Conformal Field Equations

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Conformal geometry is more fundamental than a Riemannian one. Whereas Riemannian geometry determines lengths and angles, a conformal one determines only angles and ratios of length. Equivalently, conformal geometry of space-time determines light cones, or causal structure. No length scale is a priori distinguished. It can be distinguished only a posteriori, given a particular solution of matter field equations. Einstein's field equations of gravitation can be thought of as describing interaction of causal structure with a matter described by a real scalar massless field of weight 1/4. Electromagnetic field equations need precisely a conformal structure. One can also write down field equations for a spin-1/2 Dirac massless field, given information about light cones only. The energy-momentum tensor density is obtained by vierbein variations.

1. CONFORMAL STRUCTURE OF SPACE-TIME

Let $M$ be a smooth, 4-dimensional manifold, thought of as being a model of space-time. Let $B(M)$ be the bundle of linear frames over $M$. Then $B(M)$ is a principal bundle with the structure group $GL(4)$. Let $G$ be a Lie subgroup of $GL(4)$. A $G$ structure on $M$ is a smooth subbundle of $B(M)$, with structure group $G$. In many interesting cases the structure group can be described as a group leaving invariant some tensorial object on $R^4$. For example, to give $M$ a pseudo-Riemannian structure is to give it an $O(1,3)$ structure, and $O(1,3)$ is a subgroup of $GL(4)$ leaving invariant the standard metric tensor $\gamma = \gamma_{ab} = \text{diag}(1, -1, -1, -1)$. Similarly, to give $M$ a conformal structure (plus orientation) is to give it a $CO_+(1,3)$ structure, and $CO_+(1,3)$ is the group of all $A \in GL(4)$ leaving invariant (pseudo)tensor

$$\gamma_{ab} = \frac{1}{2} \varepsilon_{cde} \eta^{ea} \eta^{fb} \quad (1.1)$$

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Let $P$ be a conformal structure on $M$. The frames in $P$ are called conformal frames (of the first order). According to a general theorem (see, for example, Kobayashi, 1972), $P$ is integrable (flat) iff each point of $M$ admits a coordinate neighborhood, with local coordinates $x^0, \ldots, x^3$, with respect to which the components of $\chi$ coincide with the standard ones [equation (1.1)]. The tensor $\chi$ is nothing but a Hodge $\ast$ operator restricted to 2-forms. Therefore, modulo topological subtleties, to give $M$ a conformal structure is to give it a smooth $\ast$ operation acting linearly on the bundle of 2-forms, and satisfying (i) $\ast^2 = -I$, and (ii) $\ast F \wedge \overline{F} = F \wedge \ast \overline{F}$ (Jadczyk, 1978). Though $\chi$ determines conformal structure completely, it is much more convenient to deal with a tensor density $\gamma_{mn}$ uniquely defined by

$$\chi_{pq}^{nm} = \frac{1}{2} \epsilon_{pqrs} \gamma^r_m \gamma^s_n = -\frac{1}{2} \gamma_{pr} \gamma_{qs} \epsilon^{rmn}$$  \hspace{1cm} (1.2)

(Gürsey, 1963, makes use of $\gamma$ in his "reformulation of general relativity in accordance with Mach's principle".) An equivalent definition of $\gamma$ can be described as follows: let $(x^m)$ be a local coordinate system, and let $(E_a)$ be a local section of the bundle of conformal frames (conformal vierbein field). Let $E_a = E_a^m \partial_m$, where $\partial_m$ are the tangents to coordinate lines. Define then

$$\gamma^{-1} = \gamma^{mn} = |\det E|^{-1/2} E^{-1}(E)$$  \hspace{1cm} (1.3)

Then $\gamma_{mn}$ is, in fact, independent of $E$, and is a symmetric tensor density of weight $W(\gamma_{mn}) = -\frac{1}{2}$, and $\det \gamma = -1$. As it was above, a conformal structure is flat iff there are local coordinate systems in which $\gamma_{mn} \equiv \eta_{mn}$. As is well known, a necessary and sufficient condition for the existence of such a coordinate system is that the Weyl conformal curvature tensor (which can be expressed in terms of $\gamma_{mn}$ only) vanishes identically.

Assume now that a conformal structure $\gamma$ is given. Let $\varphi$ be a scalar density of weight $W(\varphi) = 1/4$. Then $g_{mn} := \varphi^2 \gamma_{mn}$ is a metric tensor on $M$. In this way one gets a correspondence between conformal structures and classes of conformally equivalent Riemannian metrics. It follows, in particular, that each scalar density of weight $1/4$ determines a symmetric affine connection preserving the conformal structure. However, contrary to the Riemannian case, no such an affine connection is distinguished.

2. CONFORMAL FIELD EQUATIONS

Usually a field equation is said to be conformally invariant if it refers to the flat conformal structure of the Minkowski space, and is invariant under the whole 15-parameter group of local conformal automorphisms. To check whether this holds one has to specify transformation properties of the field under these automorphisms. It is then usually possible to deduce from these