General Relativity and General Lorentz-covariance

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Abstract

The principle of general relativity means the principle of general Lorentz-covariance of the physical equations in the language of tetrads and metrical spinors. A general Lorentz-covariant calculus and the general Lorentz-covariant generalisations of the Ricci calculus and of the spinor calculus are given. The general Lorentz-covariant representation implies the Einstein principle of space-time covariance and allows the geometrisation of gravitational fields according to Einstein's principle of equivalence.

1. The Meaning of General Lorentz-Covariant Derivatives

To understand the principle of general relativity it is necessary to distinguish between systems of coordinates \( \{x^i\} \) and systems of reference \( \Sigma' \): the former are quite mathematical and describe mathematical relations. The independence of physical quantities of the choice of the coordinate system is a logical necessity, but implies no physical consequences. On the other hand systems of reference have physical reality; they correspond to an arrangement of measurements, which determine the physical quantities.

In the simplest case such a system is realised by three measuring rods and one normal clock. To every event of the space-time \( V_4 \) three rods and a clock are attached (Treder, 1966).

A system of reference \( \Sigma \) is represented mathematically by a field of four vectors \( h^A \), which can be assumed to be orthonormalised:

\[
g_{ik} = h^A_i h^B_k \eta_{AB}, \quad \eta_{AB} = h^A_i h^B_k g_{ik}
\]  

(1.1)

Here, \( g_{ik} \) means the metric tensor of the space-time,

\[
\eta_{AB} = \text{diag}(-1, -1, -1, +1)
\]

the Minkowski tensor; the Latin minuscules are tensorial indices in the space-time and the Latin capitals deneumerate the vectors, both indices run from 1 to 4. The \( h^A_i \) themselves are functions of the space-time-coordinates \( x^I \)

\[
\frac{\partial h^A}{\partial x^I} \frac{\partial}{\partial t} h^A_{t.I} \neq 0
\]
Equation (1.1) is the condition for compatibility of the system of reference with a space-time of given metric \( g_{ik} \). A coordinate transformation from the *Einstein* group

\[ x^i = x^i(x^k), \quad h^A_i = \frac{\partial x^k}{\partial x^i} h^A_k \]  \hspace{1cm} (1.2)

transforms the tetrad vectors corresponding to the metric

\[ g_{mn} = \frac{\partial x^k}{\partial x^m} h^A_k \frac{\partial x^l}{\partial x^m} h^A_l = \frac{\partial x^k}{\partial x^m} \frac{\partial x^l}{\partial x^m} g_{kl} \]  \hspace{1cm} (1.3)

Equation (1.1) assigns universal *Minkowski* tangent space \( M_4 \) to the space-time \( V_4 \). This \( M_4 \) represents the manifold \( V_4^+ \) dual to \( V_4 \). The transformation matrix \( h^A_i \), connecting \( V_4 \) and \( V_4^+ \), is anholonomic in general with the *Einstein* object of anholonomy (Einstein, 1928; Schouten, 1953):

\[ \Delta^i_{kl} = \frac{1}{2} h^A_i (h^A_{k,1} - h^A_{l,1}) \]  \hspace{1cm} (1.4)

the \( h^A_i \) being a system of reference compatible with the given metric \( g_{ik} \). Also, the tetrad \( h^A_i \) (Lorentz-rotated in the *Minkowski* space \( V_4^+ \)) represents a compatible system of reference:

\[ h^B_i = \omega^B_A h^A_i \]  \hspace{1cm} (1.5)

with

\[ \omega^C_A \omega^B_C = \eta_{AB} \]  \hspace{1cm} (1.6)

The principle of general relativity now requires the equivalence of all systems of reference compatible with the given metric structure \( g_{ik} \) of the space-time.

This supposition is realised by the geometric objects of the space-time \( V_4 \) iff these depend on the *Lorentz* invariant combination (1.1) of the tetrads and its derivatives only (see later).

The measured values \( \phi^T \) of physical quantities are invariant with regard to choice of coordinate system, they have to be pure functions of the point and therefore scalars in the space-time:

\[ \phi^T = \phi^T(x^i(x'^i)) = \phi^T(x^i) \]  \hspace{1cm} (1.7)

Corresponding to the principle of relativity, the relations between measured values of physical quantities also have to be independent of the system of reference chosen. Especially, we have to require that \( \tilde{\phi}^T = 0 \) iff \( \phi^T = 0 \). From this it follows that the matrix has to behave covariantly with respect to the *Lorentz* transformations (1.6), i.e. the matrix of the measured values \( \phi^T \) has to be a *Lorentz* tensor of any degree:

\[ \phi^{A_1 \cdots B_1 \cdots} = \omega^{A_1 c_1 \cdots} \omega^{B_1 p_1 \cdots} \phi^{c_1 \cdots p_1 \cdots} \]  \hspace{1cm} (1.8)

From (1.7) and (1.8) it follows by (1.1) that a space-time tensor of the