We prove that we can explicitly construct the expression for a low-dimensional Hamiltonian system where proving the existence of a Smale horseshoe is equivalent to proving that Fermat's Conjecture is true. We then show that some sets of similar intractable problems are dense (in the usual topology) in the space of all dynamical systems over a finite-dimensional real manifold.

1. INTRODUCTION

We have recently started an exploration within physics and other axiomatized sciences of metamathematical phenomena such as undecidability and incompleteness (da Costa and Doria, 1991a–e, 1993; da Costa et al., 1990, 1992a,b; Stewart, 1991a,b). The original motivation for our results was a question raised by Hirsch (1985) on the (apparently) enormous difficulty of deciding whether dynamical systems which represent actual physical systems are chaotic. Hirsch asked for a general criterion to settle that question: "A major challenge to mathematicians is to determine which dynamical systems are chaotic and which are not. Ideally one should be able to tell from the form of the differential equations."

We proved (da Costa and Doria, 1991c–e) that there is no such computable criterion; chaos is algorithmically undecidable in the general situation. Actually undecidability appears everywhere in mathematics, and is a quite commonplace phenomenon; a recent example concerns the
Nonrecursivity of certain functions in algebraic geometry (Nabutovsky, 1989). It was conjectured by Wolfram (1984) that undecidability and incompleteness were also to be expected everywhere in physics: "One may speculate that undecidability is common in all but the most trivial physical theories. Even simply-formulated problems in theoretical physics may be found to be provably insoluble."

We showed that such is the case. Moreover, the chief aspect of our results is their wide-ranging applicability, as they provide a blueprint for the construction of Gödel-like incompleteness theorems within any matematized science that handles its objects through the language of classical analysis. We prove (da Costa and Doria, 1991d) that given any nontrivial property $\phi$ of a dynamical system within the language of classical elementary real analysis, we can explicitly obtain the formal expression for a countably infinite family of such systems where $\phi$ is algorithmically undecidable. Also there is another countably infinite family of systems of which it is true (in a "natural" interpretation) that they satisfy $\phi$, but such that this fact cannot be proved from the usual axiomatizations for classical analysis.

We can immediately apply that to classical mechanics. Thus, there are infinitely many expressions for infinitely many different classical mechanical systems such that one cannot prove (within a "nice" axiomatization for our theory) that those systems have a nontrivial property $\phi$, while it is true that they do have that property in all standard models, i.e., those where formalized arithmetic is interpreted as the intuitive theory of the natural numbers.

A consequence of our results is the existence of solvable but intractable problems within the realm of classical mechanics, that is to say, problems that can be stated in our formal language with the help of a small sequence of symbols, but such that their proofs are inordinately long and difficult (da Costa and Doria, 1991d; Ehrenfeucht and Mycielski, 1971; Gödel, 1986). Fermat's Conjecture that there are no positive integers $x, y, z, m$, where $x, y, and z > 1$, and $m > 2$, such that

$$x^m + y^m = z^m$$

seems to qualify (within number theory) as one of those problems with a simple statement and a very difficult proof. Well, nobody knows whether Fermat's Conjecture does, in fact, have a proof (or a counterexample), as it may also be undecidable within formalized arithmetic: if Fermat's Conjecture is false in the standard model for arithmetic, then its falsity can be proved within formalized arithmetic, but if it is true in the standard model, then it can either be provable or undecidable. If Fermat's Conjecture does have a proof, then the consensus is that it will be inordinately