The Geometry of Generalized Quantum Logics

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Received May 22, 1978

Let \( \Pi \) be a quantum logic; by this we mean an orthocomplemented, orthomodular, partially ordered set. We assume that \( \Pi \) carries a sufficiently large collection \( \Delta \) of states (probability measures). Then, \( \Delta \) is embedded as a base for the cone of a partially ordered normed space \( \mathcal{F} \) and \( \Pi \) is also embedded in the dual order-unit Banach space \( \mathcal{F}^* \). We consider conditions on the pairs \( (\Delta, \Pi) \) and \( (\mathcal{F}, \mathcal{F}^*) \) that guarantee that \( \Pi \) is a dense subset of the extreme points of the positive part of the unit ball of \( \mathcal{F}^* \). We demonstrate a connection of these conditions in noncommutative measure theory. The assumptions made here are far weaker than the assumptions of the traditional quantum mechanical formalisms and also apply to situations quite different from quantum mechanics. Finally, we show the connections of this theory to the well-known models of quantum mechanics and classical measure theory.

1. INTRODUCTION

The theory of empirical logics, or generalized quantum logics, has recently been introduced and developed by Foulis and Randall (1972, 1973a, b). In these papers, they introduced the notion of "a manual of operations," which is a formal mathematical structure describing very general laboratory procedures. Using these manuals they define the concepts of state, observable, proposition, logic, etc. These physical situations are thus very general and contain as special examples classical Boolean logic and Hilbert space quantum mechanics. More importantly, this theory contains many examples that are very different from quantum mechanics; yet, in these examples simultaneity of measurement and uncertainty can be described with reasonable proficiency. Since the results in this paper concern the geometric and linear topological properties of quantum logics embedded in certain Banach spaces, we will not begin with the foundation concept of
a manual. Rather, we will assume at the outset that our logic of propositions is given to us and that a sufficiently large collection of probability measures (states) on this logic is also provided. The reader is urged to look at the papers of Randall and Foulis already mentioned to see how the notions of logic and state are obtained from the more fundamental notion of a manual.

Let us now state the problem we are interested in describing in this paper. Let $\Pi$ denote our logic and $\Delta$ our selected convex set of probability measures on $\Pi$. All the definitions and algebraic assumptions for $\Pi$ and $\Delta$ that are necessary for the following discussion will be given with precision in the next section. Let $\mathcal{S}$ be the linear space of real-valued functions on $\Pi$ spanned by $\Delta$. Then $\Delta$ is a base for the positive cone of $\mathcal{S}$ and $\mathcal{S}$ is a base-normed space. The logic $\Pi$ can be embedded in $\mathcal{S}^*$, the partially ordered, order-unit Banach space dual to $\mathcal{S}$. The problem is this: Under what simple mathematical conditions can we identify $\Pi$ with the extreme points of the positive part of the unit ball of $\mathcal{S}^*$? Let us now discuss the problem in various situations for which the solution is known.

Let $\mathcal{H}$ be a separable Hilbert space; $\Pi$ is the orthomodular lattice of projections on $\mathcal{H}$ and $\Delta$ is taken to be the convex set of positive trace class operators with unit trace, see e.g., Jauch (1968), Chapter 8. $\mathcal{S}$ is the real linear space of self-adjoint trace class operators and $\mathcal{S}^*$ is identified with the order-unit Banach space of bounded self-adjoint operators on $\mathcal{H}$. Kadison (1951, p. 328) proved that the set of extreme points of the positive part of the unit ball of $\mathcal{S}^*$ is the set of projections on $\mathcal{H}$. In fact, he proved much more, namely, that the set of idempotents in a von Neumann algebra forms the extreme boundary of the positive part of the unit ball in the algebra. This example will be explained more fully in the last section in this article. Recently Alfsen and Shultz (1974), constructing a geometric spectral theory, proved that the projective units of a certain order-unit Banach space correspond to the extreme points of the positive part of its unit ball. This includes Kadison's result when these projective units are identified with the algebra's idempotents; further, this theory also applies to general Jordan–Banach algebras. These latter algebras need not arise from algebras of operators on a Hilbert space. Each of these examples has one common feature, namely, each element of the positive cone (of the $W^*$ algebra or the order-unit space) is represented by a spectral integral where the range of the measure defined in the integral is contained in the extreme points of the positive part of the unit ball. As we know in the standard quantum mechanical formalism, the bounded observables (positive linear functionals on the normal states) are identified with the spectral measures. If we now assume the quantum logic is finite and some other mild hypotheses are satisfied, Rüttimann (1977) has been able to identify the logic with the extreme points of the positive