Abstract. We study the Brans-Dicke vacuum field equations in the presence of a cosmological term $\Lambda$. Considering a Friedmann-Robertson-Walker metric with flat spatial sections ($k = 0$), we provide a qualitative analysis of the solutions and investigate its asymptotic properties. The general solution of the field equations for arbitrary values of $w$ and $\Lambda$ is obtained.

1. Introduction

Recently a renewed interest seems to exist in the so-called scalar-tensor theories of gravitation of which the Brans-Dicke theory (BDT) is notoriously the most investigated. Part of this interest may be attributed to the recognition of the important role these theories are able to play in the development of the contemporary models of the Universe, such as the extended inflationary cosmology program (La and Steinhardt, 1989). Another example of this interest comes from Supergravity via the mechanism of space-time dimensional compactification which generates in a rather natural way the Brans-Dicke scalar fields (Sherk, 1981).

In this paper we consider the Brans-Dicke theory of gravity (Brans and Dicke, 1961) with cosmological constant in the absence of matter. BDT solutions with a nonvanishing cosmological term have been already studied in different contexts (Uehara and Kim, 1982; Cerveró and Estévez, 1983; Lorenz-Petzold, 1984; Pimentel, 1984).

As we shall see in the next section, if we adopt the hypothesis that space is homogeneous and isotropic, then the field equations are reduced to a plane autonomous dynamical system. Therefore, it is possible to carry out a global analysis of the solutions without solving analytically the differential equations. Thus, our first task, before trying to get explicit solutions, will consist mainly of constructing the so-called phase diagrams of the system. Once obtained, the diagrams give us almost all informations about the dynamics of the models, the complete knowledge being provided by working out the general solution.

2. The Field Equations

The Brans-Dicke vacuum field equations with a nonvanishing cosmological term $\Lambda$ are given by

$$R_{\mu\nu} = -2\Lambda[(w + 1)/(2w + 3)]g_{\mu\nu} + (w/\phi^2)\phi_{,\mu} \phi_{,\nu} + (1/\phi)\phi_{,\mu} \phi_{,\nu} ,$$

$$\Box \phi = 2\Lambda\phi/(2w + 3) ,$$

* Work supported by CNPq (Brazil).
where \( w \) is the scalar field coupling constant (see, for example, Uehara and Kim, 1982).

Considering a Friedmann–Robertson–Walker metric with flat spatial section \( (k = 0) \) in the form

\[
ds^2 = dt^2 - R^2(t) [d\chi^2 + \chi^2 (d\Theta^2 + \sin^2 \Theta \, d\Phi^2)],
\]

the above equations reduces to

\[
\theta = - \theta^2/3 - (w + 1)\psi^2 - \dot{\psi} + 2\Lambda(w + 1)/(2w + 3), \quad (2a)
\]

\[
\dot{\theta} = - \theta^2 - \psi \theta + 6\Lambda(w + 1)/(2w + 3), \quad (2b)
\]

\[
\dot{\psi} = - \psi^2 - \psi \theta + 2\Lambda/(2w + 3); \quad (2c)
\]

where \( \theta = 3\dot{R}/R \) describes the expansion of the model; \( \psi = \dot{\phi}/\phi \), the overdot denoting time derivative; and \( \phi \), due to spatial homogeneity, is supposed to be a function of \( t \) only. Since in BDT the scalar field is identified to \( G^{-1} \), then \( \psi = - \dot{G}/G \) is actually a measure of the time variation of the Newtonian gravitational 'constant' \( G \).

Now, these equations lead to an algebraic relation between the variables \( \theta \) and \( \psi \):

\[
\theta^2/3 + \theta \psi - w\psi^2/2 = \Lambda, \quad (3)
\]

which can be regarded as a constraint of the dynamical system formed by any chosen pair of the set of Equations (2). In this way, let us choose (2b) and (2c) as defining our planar autonomous dynamical system.

### 3. The Equilibrium Points

The curves which appear in the phase diagrams represent the parametric solutions \( \theta = \theta(t), \, \psi = \psi(t) \) evolving in time. Generally speaking, it may happen that the dynamical system contains equilibrium points, i.e., constant solutions \( \theta = \theta_0, \, \psi = \psi_0 \), which are the roots of the right-hand side of Equations (2b) and (2c). It turns out that the equilibrium points of the system (2b)–(2c) are given by

\[
\theta_0 = 3(1 + w)\psi_0, \quad (4a)
\]

\[
\psi_0 = \pm \sqrt{2\Lambda/[(2w + 3)(3w + 4)]}. \quad (4b)
\]

In this paper let us assume from the outset that \( \Lambda > 0 \). We shall return to this point later on with brief comments on the cases \( \Lambda = 0 \) and \( \Lambda < 0 \). Thus, if \( -\frac{4}{3} < w < -\frac{4}{3} \) we do not have equilibrium points.

One should note that \( \theta_0 \) and \( \psi_0 \) also satisfy the constraint equation (3) and correspond, after straightforward integration, to de Sitter's type of solutions

\[
R(t) = R_0 \exp [(1 + w)\psi_0 t], \quad (5a)
\]

\[
\phi(t) = \phi_0 \exp [\psi_0 t]. \quad (5b)
\]

Incidentally, if \( w \to \infty \), we see that \( \psi_0 \) and \( \theta_0 \) tend to zero and \( \pm \sqrt{3\Lambda} \), respectively;