Abstract. Raychaudhuri-type equations are derived for cosmological models filled with a perfect fluid and obeying the Brans–Dicke equations with a cosmological term depending on the scalar field. In addition, some general results on spatially homogeneous cosmological models are obtained in the theories due to Bergmann and Wagoner and Uehara and Kim.

1. Introduction

After the cosmological term $A$ was introduced into general relativity by Einstein in 1917 its significance was studied by various cosmologists from time to time (cf. Petrosian, 1974; North, 1965), but no satisfactory results of its meaning have yet been reported. Zel'dovich (1968), Linde (1974), Fujii (1974), and Dreitlein (1974) have interpreted the term in different contexts. In quantum theory the assumption $A \neq 0$ means that empty space creates a gravitational field and corresponds to an energy density $\rho_A = C^4/8K$ for the vacuum. At present grand unification theories of nature predict an enormous cosmological constant (Duff, 1980). Therefore, the words of Zel'dovich and Novikov (1971) seem particularly appropriate: "After a genie is let out of a bottle (i.e., now that the possibility is admitted that $A \neq 0$), legend has it that the genie can be chased back only with great difficulty". In this paper, however, we are concerned with the theory of gravitation at the classical level. In cosmology the term may be understood by incorporation with Mach's principle, which suggests the Brans–Dicke Lagrangian as a realistic case (Brans and Dicke, 1961). Endo and Fukui (1977) have studied the Brans–Dicke cosmology with cosmological term $A$ as a function of scalar field $\phi$ and applied it to Robertson–Walker models. Recently the modified Brans–Dicke field equations for an anisotropic cosmological model have been solved by Singh and Rai (1982) and the behaviour of the model has been discussed.

In this note we propose to write Raychaudhuri-type equations for the modified Brans–Dicke theory and as a particular case for Brans–Dicke cosmology with the cosmological constant discussed by Uehara and Kim (1982). We prove general theorems for both modified Brans–Dicke cosmologies.
2. Homogeneous Cosmological Model in the Modified Brans–Dicke Cosmology

The field equations in the modified Brans–Dicke cosmology (cf. Endo and Fukui, 1977; Bergmann, 1968; Wagoner, 1970) are obtained from the variational principle

\[ \delta \left[ \phi (R - 2A(\phi)) + 16L_m - \frac{\Omega}{\phi} (\phi,\mu \phi,\nu) \right] \sqrt{-g} \, d^4x = 0, \]  

(1)

where \( \phi,\mu = \partial \phi / \partial x^\mu \), \( \phi \) being the strength of a scalar field, and cosmological term \( A \) is a function of \( \phi \). \( R \) is the usual Ricci scalar and \( L_m \) is the Lagrangian density of matter.

The field equations are

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi}{\phi} T_{\mu\nu} + \frac{\Omega}{\phi^2} (\phi,_{\mu} \phi,_{\nu} - \frac{1}{2} g_{\mu\nu} \phi,\phi) + \]  

\[ + \frac{1}{\phi} (\phi,_{\mu\nu} - g_{\mu\nu} \Box \phi), \]  

(2)

\[ \Lambda - \phi \frac{\partial A}{\partial \phi} = \frac{4\pi}{\phi} T - \frac{(2\Omega + 3)}{2\phi} \Box \phi, \]  

(3)

where semi-colons represent covariant derivatives and \( \Omega \) is a constant related to the ratio of the tensor coupling to the scalar coupling. \( A \) is an arbitrary function of \( \phi \). In the limit \( A(\phi) = \) constant, the field equations (2) and (3) reduce to those of a recently derived set of Brans–Dicke equations with cosmological constant by Uehara and Kim (1982).

Equation (2) can also be written in the form

\[ R_{\mu\nu} = \frac{8\pi}{\phi} \left[ T_{\mu\nu} - \frac{\Omega + 1}{2\Omega + 3} T g_{\mu\nu} \right] + \frac{\Omega}{\phi^2} (\phi,_{\mu} \phi,_{\nu} + \frac{1}{\phi} \phi,_{\mu\nu} + \frac{g_{\mu\nu}}{2\Omega + 3} \left[ 2\Lambda(\Omega + 1) + \phi \frac{\partial A}{\partial \phi} \right], \]

Now we take the energy momentum tensor of a perfect fluid as

\[ T_{\mu\nu} = (\rho + p) u_\mu u_\nu - pg_{\mu\nu} \quad u_\mu u^\mu = 1, \quad \rho > 0, \quad p > 0. \]  

(5)

Since \( T_{\mu\nu} = 0 \) here as in Einstein theory and Brans–Dicke theory, we must have (cf. Banerjee, 1968, 1974)

\[ i^\mu = \frac{p, h^\mu}{(p + p)}, \quad h^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu, \]  

(6)