STABILITY OF THE PLANETARY THREE-BODY PROBLEM

I. Expansion of the Planetary Hamiltonian

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Abstract. We present a direct method for the expansion of the planetary Hamiltonian in Poincaré canonical elliptic variables with its effective implementation in computer algebra. This method allows us to demonstrate the existence of simplifications occurring in the analytical expression of the Hamiltonian coefficients. All the coefficients depending on the ratio of the semi major axis can thus be expressed in a concise and canonical form.

Key words: Planetary motion, perturbation function, celestial mechanics, computer algebra.

1. Introduction

Since the early work of Laplace and Lagrange, there has been many improvements on the perturbative treatment of the long term evolution of the solar system. Among the leading works, one can quote the extended researches of Le Verrier (1855) who made an expansion of the planetary disturbing function which is still in use. It should also be stressed that until very recently, all the efforts in the perturbative treatment of the long time evolution of the solar system were devoted to the establishment of proofs of stability, although since Poincaré, the possibility of a non stable Solar system was considered. This search culminated with the results of Arnold (1963) on the existence of quasi periodic solutions in planetary problems, and also more recently with the additional stability results related to Nekhoroshev theorem.

Now, the numerical experiments (Sussman and Wisdom, 1988, Laskar, 1989, 1990a) have revealed that the solar system, and more specifically the inner solar system, is chaotic. Moreover, for the inner planets, the chaotic behavior is so extended that it excludes the possibility of application of any of the above theorems (Laskar, 1994a). This does not mean that perturbative methods should be avoided in the study of the dynamics of the planetary systems. First of all, many results on the long term behavior of the solar system were obtained using an analytical averaging of the short periods relying on perturbative methods. But also, it becomes clear that for planetary systems more reduced than our full solar system, and even for the system composed of the 4 large planets of our solar system, a perturbative approach can probably lead to stability results of some kind. These results may be obtained using KAM theory, or more specifically Arnold's theorem for the
planetary degenerated case (Arnold, 1963), and on the other hand theorems related
to Nekhoroshev theory (see for example the work of Niederman, 1994). More
numerical results on stability could also rely on the method of frequency analysis
for the estimate of the diffusion of the orbits in the phase space (Laskar, 1990a,
1993, Dumas and Laskar, 1993). In all cases, the expression of the problems
in Hamiltonian formulation will be more suitable than the usual non canonical
formulation which was adopted in the pioneered works of Laplace (1784) and
Le Verrier (1856). In the more recent works of (Laskar, 1985, 1986, 1988, 1989,
1990a), non canonical variables were also used, following the tradition of the
Bureau des Longitudes, and also in order to be able to make refined comparisons
with previous works (Bretagnon, 1974, 1982, Duriez, 1979).

In Poincaré's *Méthodes Nouvelles de la Mécanique Céleste* (1892), for the
simplicity of the demonstrations, the expressions of the disturbing function are
given in Hamiltonian form, using Jacobi coordinates, but this leads to cumbersome
expressions when one wants to practically make the computations. In later works
(Poincaré, 1896, 1897), Poincaré, recommended the use of a new set of variables,
the canonical heliocentric variables, which we have adopted here because they lead
to very elegant and symmetrical formulations of the disturbing function.

The present paper is the first of a series dedicated to the study of the stability of
planetary systems, and more specifically of the study of the planetary three body
problem. The object of these papers is also to establish a firmer ground for general
studies in planetary systems. Indeed, we believe that such analytical works will
be renewed by the study of the global stability of our solar system in connection
with the understanding of its formation, and also by the possible discovery of some
extra-solar planets in the near future. For the sake of clarity and completeness, we
have included here some material which appeared previously in some preliminary
notes.

The application of Arnold theorem to planetary systems, which will be the
subject of the joined paper (Robutel, 1995), necessitates completely analytical
expressions for the planetary Hamiltonian. The expressions given by Le Verrier
(1855) are much too complicated for this task, as they are not given in explicit
form. The derivation of suitable analytical expressions in a simple and explicit
form for the planetary Hamiltonian is the subject of this first paper.

In Section 2 the canonical heliocentric variables are introduced. Section 3 is
devoted to the presentation of a method for developing the Hamiltonian in formal
series which is well adapted to computer algebra, while the explicit algorithms are
given in Section 5.

The Section 4 is the core of this paper. This reports a new results and its demon-
stration on an unexpected simplification which appears in the explicit expansions
of the planetary Hamiltonian in term of Laplace’s coefficients. This simplification
(prop. 2) allows to reduce very much the size of the explicit expansion of the
Hamiltonian in series, which is decisive for the application of Arnold’s theorem in
the general planetary case (Robutel, 1995).