Abstract. Classical Floquet theory is reviewed with careful attention to the case of repeated eigenvalues common in Hamiltonian systems. Floquet theory generates a canonical transformation to modal variables if the periodic matrix can be made symplectic at the initial time. It is shown that this symplectic normalization can always be carried out, again with careful attention to the degenerate case. The periodic modal vectors and canonical modal variables can always be chosen to be purely real. It is possible to introduce real valued action-angle variables for all modes. Physical interpretation of the canonical degenerate normal modal variables are offered. Finally, it is shown that this transformation enables canonical perturbation theory to be carried out using Floquet modal variables.

Key words: Floquet Theory, Perturbation Theory

1. Introduction

Since the work of Floquet (1883) the solution of linear time-periodic systems has been common, especially for stability calculations of periodic orbits. It is, however, very rare to continue past the calculation of the Poincaré exponents to actually construct the periodic modal vectors. This has been done by Wiesel (1981) for the lunar theory and the Jovian moon problem. The first author also has been involved in considerable work using the periodic modal vectors in control theory (e.g.: Wiesel and Shelton (1983), Calico and Wiesel (1984)). However, none of these efforts employed a canonical version of Floquet theory.

This paper continues and corrects an earlier work, Wiesel (1980), which attempted to employ Floquet's solution as the basis of a perturbation theory. That work is chiefly defective in the failure to appreciate what was necessary to construct a canonical version of Floquet theory. Floquet theory as a canonical transformation is the subject of this paper.

2. Classical Floquet Theory

A set of canonical equations of order $2N$ can be written

$$
\dot{X} = Z \frac{\partial H}{\partial X},
$$

where $H(X, t)$ is the Hamiltonian function, and $X^T = (q_i, p_i)$ is the phase space state vector. The matrix $Z$ is

$$
Z = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix},
$$

(1)
(2)
where each identity matrix $I$ is of order $N$. We note without proof the easily established facts that $Z^T = Z^{-1} = -Z$.

The canonical variational equations then assume the form

$$\dot{x} = Z \frac{\partial^2 H}{\partial x^2} x. \tag{3}$$

We will use capital $X$ for the canonical state vector of the system (1), and lower case letters $x$ for coordinates on the tangent space of the system (1). Then a general solution close to the periodic motion can be written as

$$x(t) = \Phi(t, t_o)x(t_o), \tag{4}$$

where $\Phi(t, t_o)$ is a fundamental matrix. The fundamental matrix obeys

$$\dot{\Phi} = Z \frac{\partial^2 H}{\partial x^2} \Phi, \tag{5}$$

and (5) has the conventional initial condition $\Phi(t_o, t_o) = I$. Note that the second partials matrix $\frac{\partial^2 H}{\partial x^2}$ is a symmetric matrix, and is evaluated along the nominal trajectory $X(t)$. If the original system possesses a periodic orbit, then (3) or (5) represent a set of time-periodic linear differential equations. Also, we note that the variational equations themselves arise from

$$H = \frac{1}{2} x^T \frac{\partial^2 H}{\partial x^2} x, \tag{6}$$

which we will term the variational Hamiltonian.

Classical Floquet theory decomposes the fundamental matrix in the form

$$\Phi(t, t_o) = F(t) \exp \{ J(t - t_o) \} F^{-1}(t_o), \tag{7}$$

where $F(t)$ is periodic and $J$ is a Jordan normal form. Proofs of the existence of this form are common; in this section we review the standard techniques for constructing the solution (7).

Let the period of the periodic motion be $\tau$. The Floquet solution is constructed by first calculating the monodromy matrix $\Phi(\tau + t_o, t_o)$. This usually involves numerical integration of (1) and (5) in parallel for one period. Then since $F(t)$ is periodic, $F(\tau + t_o) = F(t_o)$, and (7) becomes

$$\Phi(\tau + t_o, t_o) = F(t_o) \exp \{ J \tau \} F^{-1}(t_o). \tag{8}$$

This directly states that $F(t_o)$ is the eigenvector matrix of $\Phi(\tau + t_o, t_o)$. If the eigenvalues of the monodromy matrix are $\lambda_i$, then the Poincaré exponents $\omega_i$, the diagonal entries in $J$, are given by

$$\omega_i = \frac{1}{\tau} \ln \lambda_i. \tag{9}$$