A Note on Debye Potentials for Spherically Gyrotropic Media

J. K. Frąckowiak and S. Przedziecki

Institute of Fundamental Technological Research, Polish Academy of Sciences, Świętokrzyska 21, PL-00-049 Warsaw, Poland

Received 22 July 1981/Accepted 15 January 1982

Abstract. The Debye potentials are generalized to the case of electromagnetic fields in spherically gyrotropic media. A medium is called spherically gyrotropic if it is locally gyrotropic with the distinguished axis having a radial direction determined by a central point. Expressions for electromagnetic fields in terms of the generalized potentials are presented and the system of differential equations for the potentials is derived. The results are summarized in the form of a theorem. Basic facts about the Debye potentials in isotropic media are recalled.

PACS: 41

The scalar Hertz potentials have been generalized recently [1] to the case of electromagnetic fields in gyrotropic media. The basic property underlying this generalization is the preservation of the rotational (axial) symmetry of the Maxwell system for the gyrotropic tensors ε, μ.

As is well known, for isotropic media the spherical symmetry of Maxwell's equations makes possible the introduction of another pair of auxiliary functions called the Debye potentials [2] or the spherical Hertz potentials [3].

One easily notes that this symmetry remains preserved if the constitutive tensors are locally gyrotropic with the distinguished axis having a radial direction defined by a fixed point and with the properties of the medium depending, at the most, on the distance from the considered point.

This observation suggests a possibility to generalize the Debye potentials to the case of electromagnetic fields in spherically symmetric anisotropic media. This generalization is the purpose of the present paper.

The general forms of spherically symmetric tensors ε, μ in an appropriate system of spherical coordinates (θ, φ, r) are

\[
\begin{pmatrix}
\varepsilon & -i\varepsilon_g & 0 \\
-i\varepsilon_g & \varepsilon & 0 \\
0 & 0 & \varepsilon_r
\end{pmatrix}
\quad
\begin{pmatrix}
\mu & -i\mu_g & 0 \\
-i\mu_g & \mu & 0 \\
0 & 0 & \mu_r
\end{pmatrix}
\]

where ε and μ are, in general, functions of r. Harmonic time dependence has been assumed with the time factor \( \exp(-i\omega t) \) which will be suppressed throughout.

A medium that is described in an appropriate spherical system (θ, φ, r) by the constitutive tensors of forms (1) will be called spherically gyrotropic provided not both \( \varepsilon_g = 0 \) and \( \mu_g = 0 \). If \( \varepsilon_g = \mu_g = 0 \) but \( \varepsilon + \varepsilon_r \) and/or \( \mu + \mu_r \), the medium will be said to be radially uniaxial.

The physical relevance of spherically gyrotropic media is rather very limited. Nevertheless the generalized Debye potentials could be used advantageously in the analysis of propagation in a plasma in the vicinity of the poles of a magnetic dipole. A magneto-ionic model of the ionosphere for high and medium latitudes, widely exploited by Krasnushkin [4, 5] for investigations of propagation of very long waves, is based on the tensor ε of form (1), μ being a scalar. Also impedance boundary conditions employed by Wait [6] would follow from such a model of the ionosphere.

The generalization of the Debye potentials to the case of radially uniaxial media was presented in [7].

1. Notation

We shall denote by r the distance from a fixed point and by \( \mathbf{r}_o \) the outward unit normal to the spheres...
We shall refer to the direction $r_0$ as radial or longitudinal and to the directions tangent to the spheres $r=\text{const}$ as transverse.

We introduce the following notation for the transverse parts of the operators $\text{grad}$, $\text{div}$, and $\text{Laplacian}$:

\[
\nabla_t = \nabla - r_0 \frac{\partial}{\partial r},
\]
\[
\nabla^2_t = \nabla \cdot \nabla = \nabla_0 \cdot \frac{\partial}{\partial r} (r^2 \frac{\partial}{\partial r}).
\]

The operator $\text{curl}$ can now be written as

\[
\nabla \times = r_0 \times \frac{\partial}{\partial r} (r)
\]
\[
= -r_0 \times \nabla_t(r_0 \cdot) - r_0 \nabla_t(r_0 \times ) .
\]

2. Isotropic Medium

For isotropic media the basic facts characterizing the Debye potentials can be summarized in the following two theorems:

**Theorem 1.** An electromagnetic field $E, H$ generated in a region $D$ from two scalar functions $u, v$ via formulas:

\[
E = \nabla \times \nabla \times ru + i \omega \mu \nabla \times ru_0
\]
\[
H = -i \omega \varepsilon \nabla \times ru_0 + \nabla \times \nabla \times ru_0
\]

satisfies in $D$ the homogeneous set of Maxwell's equations:

\[
\nabla \times H = -i \omega \varepsilon \nabla E
\]
\[
\nabla \times E = i \omega \mu \nabla H
\]

if the functions $u$ and $v$ fulfill in $D$ the Helmholtz equation:

\[
(\nabla^2 + k^2) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\]

where $k^2 = \omega^2 \varepsilon \mu$.

With the aid of the transverse operators, (3) can be split into their transverse and longitudinal parts:

\[
e = \nabla_t \frac{\partial u}{\partial r} + i \omega \mu \nabla_t v \times r_0
\]
\[
h = -i \omega \varepsilon \nabla_t v \times r_0 + \nabla_t \frac{\partial v}{\partial r},
\]
\[
E_r = -\nabla^2_t ru
\]
\[
H_r = -\nabla^2_t rv,
\]

where

\[
e = E - E_0, \quad h = H - H_0, \quad E_r = E \cdot r_0, \quad H_r = H \cdot r_0.
\]

Let us also add that the choice of the central point $r=0$ is completely arbitrary.

The proof of Theorem 1 follows from a simple substitution of (3) into (4). The functions $u$ and $v$ are called Debye potentials [2] or spherical Hertz potentials [3]. Their relation to the Hertz vectors is clarified in [2]. From Theorem 1 follow two corollaries:

a) The field $E^0, H^0$ generated by the function $u$ is transverse magnetic (TM) with respect to the radial direction $r_0$, i.e. $H^0 \cdot r_0 = 0$ and the field $E^0, H^0$ generated by $v$ is transverse electric (TE) with respect to $r_0$, i.e. $E^0 \cdot r_0 = 0$.

b) Each of the fields $E^0, H^0$ and $E^0, H^0$ satisfies the system (4).

The formulas (3) show that with their aid we can generate a large class of sourceless electromagnetic fields from the set of all wave functions $u, v$ determined in $D$. A question that immediately arises is whether this class coincides with the set of all sourceless electromagnetic fields in $D$. (The point $r=0$ is considered to be fixed.)

An answer to this question is provided by the following representation theorem:

**Theorem 2 (Representation Theorem).** An arbitrary sourceless electromagnetic field $E, H$ given in a region $D$ of sufficiently simple shape can be represented in $D$ in terms of two scalar functions $u, v$ in the form (3) with the functions $u, v$ satisfying in $D$ the Helmholtz equation.

The restriction on the region $D$ to be of sufficiently simple shape means that any straight half-line of radial direction must not have more than one interval in common with $D$.

An alternative formulation of Theorem 2 can be given as the following splitting theorem:

**Theorem 2' (Splitting Theorem).** An arbitrary sourceless electromagnetic field $E, H$ given in $D$ can be split into a TM and a TE field with respect to $r_0$ so that each of these constituent fields satisfies (4) and can be expressed in terms of one scalar function in the form being the respective part of (3) with the scalar function satisfying (5). The restriction on $D$ is the same as in Theorem 2.

In both Theorems 2 and 2' the point $r=0$ can be chosen arbitrarily as long as the restriction imposed on the region $D$ is fulfilled.

A proof of Theorem 2 (or 2') for the interior or the exterior of a sphere $r=\text{const}$ is given in [3]. For more general regions a proof could be conducted along the lines similar to those adopted in [8].

Theorems 1 and 2 (or 2') can be considered as mutually inverse provided Theorem 1 is confined to regions for which Theorem 2 (or 2') holds.

Theorems 1, 2, 2' are easily generalized to the case of a radially uniaxial medium [7] but with one essential