A Propagation of Chaos Result for Burgers’ Equation

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I. Introduction

In [15], McKean posed the problem of constructing a system of $N$ interacting particles in $\mathbb{R}$ with generator

$$L = \frac{1}{2} \sum_{i} \frac{\partial^2}{\partial x_i^2} + \frac{1}{2(N-1)} \sum_{i<j} \delta(x_i-x_j) \left( \frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j} \right).$$

(1.1)

He conjectured that when the initial conditions are independent and $u_0$ distributed, and if one looks at the law at time $t$ of the first $k$ particles, $k$ fixed, letting the number $N$ of interacting particles be larger and larger, one restores asymptotically at time $t$ the independence of our first $k$ particles $(X_1^t, \ldots, X_k^t)$, and that their common limiting ($N \to \infty$), distribution is given by the value at time $t$ of the solution of Burgers’ equation:

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} - u \frac{\partial u}{\partial x}, \quad \text{with initial condition } u_0 \text{ at time } 0.$$  

(1.2)

Such a type of phenomenon is called propagation of chaos (see Kac [12]). Several results concerning the questions of propagation of chaos and Burgers’ equation have already been obtained, in Calderoni-Pulvirenti [2], where a smoothing procedure of the $\delta$-function is used, in Gutkin-Kac [6], and Kotani-Osada [13], where the approach for the construction of the $N$-particle process and for the propagation of chaos result is rather analytical.

The approach presented here is probabilistic.

We consider a system of $N$ particles satisfying:

$$dX_i^t = dB_i^t + \frac{c}{N} \sum_{j \neq i} dL^0(X_i^t - X_j^t), \quad i = 1, \ldots, N,$$

(1.3)

$$X_0^i = X^i(0),$$

where $L^0(X_i^t - X_j^t)$ is the symmetric local time in $0$ of $X_i^t - X_j^t$, $B_i^t$ are independent Brownian motions, independent of the initial conditions $(X_i^t(0))$, with
symmetric distribution \( u_\lambda \) satisfying:
\[
\bigcup_{i,j,k \text{ distinct}} \{ x^i = x^j = x^k \} \text{ is } u_\lambda \text{-negligible.} \quad (1.4)
\]

Such a process was constructed with a probabilistic approach in Sznitman-Varadhan [21], where it is also shown that the process (1.3), is trajectorially approximated by the "smoothed" processes:
\[
d X_t^{i,\alpha} = dB_t^{i,\alpha} + \frac{c}{N} \sum_{j \neq i} \phi_\alpha \left( X_t^{i,\alpha} - X_t^{j,\alpha} \right) 2dt, \\
X_0^{i,\alpha} = X^{i}(0),
\]
when \( \alpha \) goes to zero, \( \left( \phi_\alpha (\cdot) = \frac{1}{\alpha} \phi \left( \frac{\cdot}{\alpha} \right) \right) \) is an approximation of the Dirac measure.

This stability result links (1.3) with (1.1) (take \( c = 1/4 \)), specially if one notices that (1.1) can be interpreted as the divergence type operator
\[
\mathcal{L} = \text{div}(A \text{ grad}), \quad A = \frac{1}{2} Id + \frac{H(x)}{4(N-1)}, \quad \text{if } H_{ij}(x) = H(x^i - x^j),
\]
(\( H(t) = 1(t \geq 0) - 1(t \leq 0) \)). This remark concerning the generator (1.1) was a key point noticed by Kotani-Osada [13].

Let us first introduce a

Definition. If \( E \) is a separable metric space, \( \nu \) a probability on \( E \), a sequence \( (\nu_N) \) of symmetric probabilities on \( E^N \) is said to be \( \nu \)-chaotic, if for \( \phi_1, \ldots, \phi_k \), continuous bounded functions on \( E \),
\[
\lim_{N \to \infty} \langle \nu_N, \phi_1 \otimes \cdots \otimes \phi_k \otimes 1 \otimes \cdots \otimes 1 \rangle = \prod_{i=1}^{k} \langle \nu, \phi_i \rangle. \quad (1.6)
\]
In the following \( M(E) \) will denote the set of probabilities on \( E \). One can show (see Tanaka [23], Sznitman [20]), that being \( u \)-chaotic is equivalent to
\[
\bar{X}_N = \frac{1}{N} \sum_{i=1}^{N} \xi^i \quad (\text{which is a } M(E) \text{-valued r.v. defined on } (E^N, \nu_N)), \quad (1.7)
\]
\( X_i \) are the canonical coordinates on \( E^N \), converges in law towards the constant \( \nu. (M(E) \text{ is endowed with the topology of weak convergence which allows us to define the convergence in law for the } M(E)\text{-valued sequence of r.v. } \bar{X}_N). \)

In this work we obtain the fact that for \( u_N, u \)-chaotic, \( (E = \mathbb{R}; \text{ in the previous definition}) \), and satisfying (1.4), then the laws \( P_N \), on \( C(\mathbb{R}_+, \mathbb{R})^N \) of the processes \( (X^i) \) satisfying (1.3), with initial law \( u_N \), are \( P \)-chaotic, (now \( E = C(\mathbb{R}_+, \mathbb{R}) \)), where \( P \) is the law of the nonlinear process which describes the asymptotic \( (N \to \infty) \) individual behavior of the particles. Roughly speaking, this nonlinear process is obtained by considering (1.3), for say particle 1, and