Persistent Random Walks in Random Environment

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Summary. Weak convergence of a class of functionals of PRWRE is proved. As a consequence CLT is obtained for the normed trajectory.

1. Introduction

In the present article we investigate the asymptotic behaviour of Persistent (or Physical) Random Walks (PRW) in Random Environment (RE) on \( \mathbb{Z}^d \).

Suppose \( \mathcal{U} \subset \mathbb{Z}^d \) is a finite symmetric subset of the lattice generating the group of translations on \( \mathbb{Z}^d \): the set of possible steps of the random walker. At each site of the lattice \( z \in \mathbb{Z}^d \) a random scatterer is placed characterized by a stochastic matrix \( \Gamma(z) = (\gamma_{u,u'}^{(z)})_{u,u' \in \mathcal{U}} \), which we shall call the persistency (or scattering) matrix at \( z \). The persistency matrices are random and the collection of them is the RE.

Given the environment a PRW is a Markov chain of order two on \( \mathbb{Z}^d \) with transition probabilities

\[
P(X_{n+1} = z + u' \mid X_n = z, X_{n-1} = z - u) = \gamma_{u,u'}^{(z)}.
\]

The model can be considered as a stochastic version of the Lorentz gas with finite horizon.

The following three conditions are imposed on the scattering matrices:
0. \((\Gamma(z))_{z \in \mathbb{Z}^d}\) form a stationary and ergodic sequence of random matrices (under translations on \( \mathbb{Z}^d \))
1. they are almost surely bistochastic
2. they satisfy almost surely a uniform Doeblin condition (condition (b) of the next section).

Comment: Condition 0. is natural. Condition 1. is the weakest symmetry condition on the scattering mechanism. In fact, from a physical point of view

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stronger conditions seem to be natural (e.g. $\gamma_{u,u'} = \gamma_{-u',-u}$ a.s.). Condition 2. is a technical one, and we think that it can be considerably weakened.

Under the above conditions we prove that the finite dimensional distributions of $e^{1/2} X_{[t-1]}$ converge to those of a Brownian motion with a positive definite covariance matrix (in probability with respect to the environment).

For the one-dimensional case with nearest neighbour jumps the invariance principle has been proved in [6] using a "very one-dimensional" ad hoc argument, so this result can be considered as a generalization of that one. The study of this model was proposed by D. Szász.

Besides the intrinsic physical interest of the model, we think that the main mathematical interest of our result consists of the fact that it relies on a generalization to non-reversible Markov chains of a theorem recently announced by C. Kipnis and S.R.S. Varadhan on the asymptotics of additive functionals of reversible Markov chains [3].

The paper consists of two further sections and an Appendix. In Sect. 2 we give the exact mathematical formulation of the problem and state our result. In Sect. 3 the proof is given. The Appendix contains the sketch of proof of the generalization of the Theorem of Kipnis and Varadhan.

2. Exact Formulation and Main Result

Before entering into details we have to specify some notations.

Throughout this paper $D([0, 1])$ will denote the space of right-continuous real functions defined on $[0, 1]$, endowed with the Skorohod topology. The Wiener measure of variance $\sigma^2 \geq 0$ on $D([0, 1])$ is denoted by $\mathcal{W}_\sigma$.

Let $(\eta^n_{t/n})_{n \in \mathbb{N}}$ be a Markov chain with state space $\Omega$ and trajectory space $\Omega^\mathbb{N}$ (endowed with the natural product $\sigma$-algebra). We shall denote by $\Pi^{(\omega)}$ the Markovian measure on $\Omega^\mathbb{N}$ conditioned to the initial state $\eta_0 = \omega$. If $\mu$ is a probability measure on $\Omega$ (that is: an initial distribution of the Markov chain), $\Pi^{\mu}$ will denote the Markovian measure on the space of trajectories conditioned to this initial distribution

$$\Pi^{\mu} = \int_\Omega \mu(d\omega) \Pi^{(\omega)}.$$

We say that the Markov chain $\eta_n$ is stationary (ergodic) with respect to $\mu$ if the measure $\Pi^{\mu}$ is stationary (ergodic) under the left shift of $\Omega^\mathbb{N}$. We denote by $\mathcal{P}$ the transition kernel acting on the space of bounded, measurable functions defined on $\Omega$. If $\mathcal{P}$ is compatible with the $\mu$-equivalence of measurable functions, $\mathcal{P}$ and $\mathcal{P}$ will denote the transition operator and its adjoint acting on the $L_p(\Omega, \mu)$ spaces.

The following lemma is standard (see for example [4])

**Standard Lemma**

i) Let $\rho \in L_1(\Omega, \mu)$, $\rho \geq 0$ $\mu$-a.s., $\int d\mu \rho = 1$. The Markov chain $\eta_n$ is stationary w.r.t. $\rho d\mu$ if and only if

$$\mathcal{P} \rho = \rho$$