Asymptotic Expansions Based on Smooth Functions in the Central Limit Theorem

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Summary. Stein’s method is used to derive asymptotic expansions for expectations of smooth functions of sums of independent random variables, together with Lyapounov estimates of the error in the approximation.

1. Introduction

When considering the error in the normal approximation to the partial sums of stationary sequences, Stein (1970) introduced a new technique, by means of which differences of the form $\mathbb{E}h(W) - \mathbb{E}h(N)$, for smooth functions $h$, could be directly estimated: here, $N$ denotes a standard normal random variable, and $W$ denotes the random variable whose distribution is being approximated. The technique has a structure which lends itself in principle to iterative application, by means of which asymptotic expansions could be obtained, but the possibility seems not to have been exploited, owing to the apparent complexity of the procedure. In this paper, a simplification is found, which enables asymptotic expansions for the expectations of smooth functions of sums of independent random variables to be derived, together with Lyapounov bounds on the approximation error, at the cost of some analytic argument, concerning the smoothness and rate of growth of solutions of Stein’s ordinary differential equation. The principal tool is a lemma which, for a random variable $X$ with exponentially decaying tails, makes explicit the way in which the difference between $\mathbb{E}h(X)$ and $\mathbb{E}h(N)$ depends on the cumulants of $X$ of order greater than two.

Asymptotic expansions for $\mathbb{E}h(W)$, where $h$ is smooth and $W$ is a partial sum of independent random variables, were considered by Hsu (1945), von Bahr (1965) and Bhattacharya (1970), and have more recently been discussed in Hipp (1977) and in Götze and Hipp (1978). In the latter paper, asymptotic expansions are obtained by Fourier methods, under conditions which, although similar to those used here, are not equivalent, and their error estimates are more difficult to express: however, their results are proved for sums of inde-
dependent random vectors in \( \mathbb{R}^d \). The arguments used in this paper are understandably simpler.

An advantage of considering expansions only for expectations of smooth functions \( h \) of \( W \) is that, in contrast to expansions for distribution functions, there is no need to impose smoothness conditions on the distributions of the summands: the natural moment conditions are all that is required. However, the problem cannot be entirely avoided. Each extra term in the asymptotic expansion requires an extra derivative of the function \( h \) to exist, and the estimated error of the expansion depends on a Lipschitz measure of the smoothness of the highest required derivative of \( h \).

2. Main Results

The essence of Stein's (1970) method is that, if \( h \) is any function for which \( \mathbb{E} |h(\mathcal{N})| < \infty \), then, for any random variable \( X \),

\[
\mathbb{E} h(\mathcal{N}) - \mathbb{E} h(X) = \mathbb{E} \{X g(X) - Dg(X)\},
\]

where \( g = \partial h \) is defined by

\[
(\partial h)(x) = \int_{-\infty}^{\infty} e^{\frac{1}{2}(x^2 - t^2)} \{h(t) - \mathbb{E} h(\mathcal{N})\} \, dt
\]

\[
= -\int_{-\infty}^{x} e^{\frac{1}{2}(x^2 - t^2)} \{h(t) - \mathbb{E} h(\mathcal{N})\} \, dt.
\]

Here, and subsequently, \( \mathcal{N} \) denotes a standard normal random variable and \( D_i f \) the \( i \)th derivative of \( f \). Note that \( g \) satisfies the differential equation

\[
D g(w) - w g(w) = h(w) - \mathbb{E} h(\mathcal{N}).
\]

Thus the closeness of the distributions of \( X \) and \( \mathcal{N} \) can be estimated, if an estimate of \( \mathbb{E} \{X g(X) - Dg(X)\} \) is available, for an appropriate class of functions \( g \). The following lemma provides a starting point for such estimates.

**Lemma 1.** Let \( g \) be an \( l-1 \) times differentiable function and \( X \) a random variable, and suppose that

\[
\mathbb{E} \{|X|^l u_{l-1}(g; X)\} < \infty,
\]

where

\[
u_k(g; x) = \sup_{|t| \leq x} |D_k g(t) - D_k g(0)|.
\]

Let \( \kappa_r \) denote the \( r \)th cumulant of \( X \). Then

\[
\mathbb{E} \{X g(X)\} = \sum_{s=0}^{l-1} \binom{l-1}{s} \frac{\kappa_{s+1}}{s!} \mathbb{E} \{D_s g(X)\} + \eta_{l-1}(g; X),
\]