Abstract. A comparison is made between the stability criteria of Hill and that of Laplace to determine the stability of outer planetary orbits encircling binary stars. The restricted, analytically determined results of Hill’s method by Szebehely and co-workers and the general, numerically integrated results of Laplace’s method by Graziani and Black are compared for varying values of the mass parameter $\mu = m_2/(m_2 + m_1)$. For $0 < \mu < 0.15$, the closest orbit (lower limit of radius) an outer planet in a binary system can have and still remain stable is determined by Hill’s stability criterion. For $\mu > 0.15$, the critical radius is determined by Laplace’s stability criterion. It appears that the Graziani-Black stability criterion describes the critical orbit within a few percent for all values of $\mu$.

Key words: Planetary orbits – stability.

1. Introduction

Studying the orbital stability of three-body systems is essential to various fields in astronomy, astrophysics, and aerospace engineering, including such problems as triple-star systems, perturbed satellite orbits, and planetary orbits. Although we still have no definitive evidence of a planetary system other than our own, the search for other planetary systems has gained much support in recent years. Orbital stability studies and three-body calculations can help to constrain and possibly predict the location and type of orbits we may expect to observe in such systems.

This paper is concerned with the stability of outer planetary orbits in binary star systems, where an "outer" planet is one with an orbit that encloses both primaries. Our focus is on binaries because the majority of stars occur in binary systems. The basic question concerning the stability of outer planets can be formulated as follows: for a binary system with given masses ($m_1 \geq m_2$), what is the smallest radius that an outer planet in an approximately circular orbit can have for stability? If the outer planet is far enough from the binary, the perturbation of the binary will not disturb the planet’s orbit, and it can be considered stable. If, on the other hand, the size of the outer planetary orbit is below the limit for stability, the binary would disrupt the planet’s orbit. It should be noted that if the planetary orbit is very far from the binary, its orbit might be perturbed by other members of the
galactic system, and the binary could lose the planet. This type of instability is not considered in the present paper.

Comparison of various stability investigations is a highly sensitive procedure as the results depend strongly on the definitions used for stability. In 1984, a review (Szebehely, 1984) resulted in approximately 50 different definitions and terminologies used for stability. This number today would be close to 100, indicating the increased interest in stability research.

The two concepts to be compared in this paper are associated with concepts due to Hill and to Laplace. Because many variations of their original formulations and definitions exist today, we will clarify and explain the exact approaches being compared here. Hill’s stability criterion uses the concept of zero-velocity surfaces. The stability of an outer planet depends on the value of its actual Jacobian constant as compared to the critical value of the Jacobian constant at the equilibrium point (Szebehely, 1980; Szebehely and McKenzie, 1982). Stability according to Laplace means that no secular trends develop in the orbital elements of any of the bodies during the evolution of the system (Graziani and Black, 1981; Black, 1982; Pendelton and Black, 1983). The difference between these two definitions is considerable in the outer planet case. Hill’s stability requirement is still satisfied if the planet escapes from the binary as long as it stays outside Hill’s limiting zero velocity curve.

Laplace’s concept, on the other hand, considers the escape of the outer planet as a form of instability.

We will compare the results of stability analyses for planar motions in this paper, but both approaches can be generalized to three-dimensional configurations. We will also assume circular motions for the binary. This assumption can also be generalized in the Laplace approach, but it may present some conceptual difficulties for Hill’s method which is based on the existence of the Jacobian integral (which does not exist if the members of the binary move on elliptic orbits).

2. Review of Previous Work

In 1982, a comparison was made (Black, 1982) between numerically integrated, general three-body problem (TBP) results and an analytically determined set of results for the restricted problem. The analytic results, based on Hill’s method (Szebehely, 1980; Szebehely and McKenzie, 1981), are shown in Figure 1 where \( \mu = m_2/(m_1 + m_2) \) and \( r \) is the distance between the planet and the center of mass of the binary. We are here using the following definitions: an “outer” planet is one whose orbit encircles the binary, “inner” planet refers to a planet that orbits the larger of the primaries, and “satellite” refers to a planet orbiting the smaller primary.

The results of a study using Laplace’s method (Graziani and Black, 1981; Black, 1982) are shown in Figure 2 as a solid line. This line, the Graziani-Black (GB) stability criterion, denotes the division between stable (below the line) and unstable