Foundations of Physics, Vol. 8, Nos. 5/6, 1978

Geometric Significance of the Spinor Lie Derivative. I

V. Jhangiani[1]

Received May 9, 1977

In a previous article, the writer explored the geometric foundation of the generally covariant spinor calculus. This geometric reasoning can be extended quite naturally to include the Lie covariant differentiation of spinors. The formulas for the Lie covariant derivatives of spinors, adjoint spinors, and operators in spin space are deduced, and it is observed that the Lie covariant derivative of an operator in spin space must vanish when taken with respect to a Killing vector. The commutator of two Lie covariant derivatives is calculated; it is noted that the result is consistent with the geometric interpretation of the Jacobi identity for vectors. Lie current conservation is seen to spring from the result that the operator of spinor affine covariant differentiation commutes with the operator of spinor Lie covariant differentiation with respect to a Killing vector. It is shown that differentiations of the spinor field defined geometrically are Lorentz-covariant.

1. INTRODUCTION

The student of theoretical physics often hears the statement that since a spinor has no directly geometric significance, there is no unique way of differentiating it covariantly in Riemannian spacetime. In Ref. 1, however, it was seen that the affine covariant derivative of spinor analysis has a simple geometric meaning, which is based on the assumption that parallel transfer of a spinor in Riemann space is given by parallel transfer of the underlying orthonormal basis. The same kind of reasoning is invoked here to calculate the Lie covariant derivative of a spinor in Riemann space, the assumption being made that Lie transfer of a spinor is given by Lie transfer of the underlying orthonormal basis.

In Section 2, the Lie differentiation of tensors is briefly reviewed; in Section 3 the assumption of geometric spinor Lie transfer is used to define

[1] Department of Physics, University of California at Santa Barbara, Santa Barbara, California.

0015-9018/78/0600-0445$05.00/0

the Lie derivative of a spinor. It is shown in Section 3 that the Lie covariant derivative of an operator in spin space must vanish when taken with respect to a Killing vector; the spinor Lie derivatives of the spin operators β^i and the nucleus σ^{ik} are also given. In Section 4, the commutator of two Lie covariant derivatives is calculated, and the underlying mathematical significance of Lie current conservation is investigated in Section 5. The proof of the Lorentz covariance of the rule for geometric spinor transport is given in an appendix.

2. LIE DERIVATIVE OF A VECTOR

The Lie derivative of the vector A with respect to the vector ξ is another vector defined as[2]

$$\mathscr{L}_\xi A = [\xi, A] = \xi A - A\xi \tag{1}$$

i.e.,

$$(\mathscr{L}_\xi A)^i = \xi(A^i) - A(\xi^i) \tag{1a}$$

More explicitly,

$$(\mathscr{L}_\xi A)^i = A^i{}_{,k}\xi^k - \xi^i{}_{,k}A^k \tag{2}$$

There may be ascribed to Eq. (2) a geometric significance[3,4] that can be summarized algebraically as follows. Let x^i and $x^i + dx^i = x^i + \epsilon\xi^i$, where ϵ is infinitesimal, be two points of the spacetime manifold in infinite proximity. The vector A^i can be transported from x^i to $x^i + dx^i$ according to the rule

$$\Delta A^i = \epsilon\xi^i{}_{,k}A^k \tag{3}$$

Equation (3) gives the law of *Lie transport* for the contravariant vector A^i; with the vector A^i at x there is associated the vector $(A^i + \Delta A^i)$ at $x + dx$, said to have been obtained from the former via Lie transport of A with respect to ξ.

The corresponding law of transport for the covariant vector B_i is[2]

$$\Delta B_i = -\epsilon\xi^k{}_{,i}B_k \tag{4}$$

From (3) and (4),

$$\Delta(A^i B_i) = 0 \tag{5}$$

corresponding to the invariance of a scalar under coordinate transformation. It is evident from Eq. (5) that Lie transport preserves inner products.

[2] More generally, the law of Lie transport for the tensor **T** is given by formally considering the equation $x'^i = x^i + \epsilon\xi^i(x)$ as an infinitesimal coordinate transformation, and calculating the corresponding change in the components of **T**.