Clebsch Representations and Energy–Momentum of the Classical Electromagnetic and Gravitational Fields

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By means of a Clebsch representation which differs from that previously applied to electromagnetic field theory it is shown that Maxwell’s equations are derivable from a variational principle. In contrast to the standard approach, the Hamiltonian complex associated with this principle is identical with the generally accepted energy–momentum tensor of the fields. In addition, the Clebsch representation of a contravariant vector field makes it possible to consistently construct a field theory based upon a direction-dependent Lagrangian density (it is this kind of Lagrangian density that may arise when developing the Finslerian extension of general relativity). The corresponding field equations are proved to be independent of any gauge of Clebsch potentials. The law of energy–momentum conservation of the field appears to be covariant and integrable in a rather wide class of direction-dependent Lagrangian densities.

1. INTRODUCTION

The Clebsch representations of vector and n – 2 index tensor fields on any n-dimensional differentiable manifold (n ≥ 3) have been constructed locally in Rund’s recent work. This work has also demonstrated the interesting potentialities of the application of Clebsch potentials to field theory. In particular, the interesting fact has been discovered that, in treating the electromagnetic field as a field of skew-symmetric covariant second-rank tensors, the two pairs of Maxwell’s equations in vacuo and the representation of the electromagnetic tensor as the curl of a four-potential are obtainable simultaneously from a natural variational principle. Later, it was shown (Rund, Baumeister) that the Clebsch potentials made it possible to develop a profound generalization of the Hamilton–Jacobi theory in which

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the canonical equations of motion and the equation of continuity result from a multiple variational principle involving the Clebsch potentials. It is these works that have motivated the present paper.

In Section 2, we convert the four-potential electromagnetic field theory into the Clebsch representation to demonstrate that this procedure does not affect the electromagnetic field equations in vacuo. Then the electromagnetic energy–momentum tensor is calculated in the Clebsch representation, which results in the fact that the expression known as the symmetric tensor of the energy–momentum of the electromagnetic field is directly obtainable in this way. This expression can also be obtained, if one follows Rund’s treatment of the electromagnetic field, as a field of skew-symmetric tensors.

In Section 3, we touch upon the problem of developing a field theory in which the Lagrangian density is a function not only of points \( x^i \) of an underlying manifold, but also of directions of tangent vectors \( y^i \) attached to \( x^i \). Proceeding from this kind of Lagrangian density, treatment of the \( y^i \) as additional variables essentially independent of \( x^i \) does not yield a satisfactory theory. This was argued particularly by Rund and Beare.\(^{(6)}\) Therefore, we shall make an attempt to develop an alternative approach regarding the \( y^i \) as functions of \( x^i \). Examination of this possibility is especially interesting because, apparently, the development of the Finslerian generalization of general relativity should be carried out with the help of the concept of a preferred vector field (Ref. 7, p. 228). The Euler–Lagrange equations are invariant with respect to the Clebsch potentials of the vector field \( y^i(x) \) and give Clebsch-gauge-invariant field equations \([\text{Eqs. (25)}]\) describing \( y^i(x) \). Finally, attention is paid to the interesting fact that a covariant and integrable law of conservation of the field energy–momentum can naturally be formulated in a rather wide class of direction-dependent Lagrangian densities.

We restrict ourselves to a four-dimensional manifold \( X_4 \) at least of class \( C^2 \) referred to local coordinates \( x^i; \ i, j, \ldots, = 1, 2, 3, 4 \). For the sake of brevity, we shall write \( \partial_i \) instead of \( \partial/\partial x^i \). The usual summation convention will be adopted. Let \( A_i(x) \) denote a class \( C^2 \) covariant vector field of the greatest character. Then, there will exist four independent scalar functions \( P_a(x), Q^a(x), \) \( a = 1, 2 \), such that (locally)\(^{(1–5)}\)

\[ A_i(x) = P_a \partial_i Q^a \tag{1} \]

The scalars \( P_a, Q^a \) are referred to as the Clebsch potentials of the vector field \( A_i(x) \).

The analogous representations of tensor fields of other types were described in Rund’s work.\(^{(6)}\) In particular, a contravariant vector field can be represented in the form (19), and a covariant skew-symmetric second-rank tensor field in the form (17). The transition from a vector or a tensor field to