Mutually Exclusive and Exhaustive Quantum States

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Received May 8, 1975

The identification of a set of mutually exclusive and exhaustive propositions concerning the states of quantum systems is a cornerstone of the information-theoretic foundations of quantum statistics; but the set which is conventionally adopted is in fact incomplete, and is customarily deduced from numerous misconceptions of basic quantum mechanical principles. This paper exposes and corrects these common misstatements. It then identifies a new set of quantum state propositions which is truly exhaustive and mutually exclusive, and which is compatible with the foundations of quantum theory.

1. LOGICAL SPECTRA AND STATISTICAL PHYSICS

In recent years the abstract principles of information theory have been superposed upon the fundamental laws of mechanics in order to erect the old discipline of statistical mechanics on a basis more plausible, rational, and systematic than has historically been the case. These efforts must be regarded as very successful, if for no other reason than that they have clarified better than ever before just what really are the essential foundations of statistical mechanics. Nevertheless, the present state of those foundations remains shaky in the realm of quantum statistics because quantum mechanics itself is beset by numerous controversial misunderstandings. As we shall see in detail below, many of these common misinterpretations of quantum physics have been absorbed uncritically into the fabric of information-theoretic statistical mechanics, with the result that quantum statistics is not yet truly as well grounded as a cursory survey might suggest.

To apply information theory to any situation, the first step consists
in identifying a list of relevant propositions concerning that situation. The list must be *mutually exclusive* (no two propositions can simultaneously be true) and *exhaustive* (one of the propositions is certainly true). Such a set of mutually exclusive and exhaustive propositions has sometimes been called a *logical spectrum*, and we shall, for brevity, adopt that terminology.\(^{(1)}\)

Suppose now that we confront in a physical laboratory a complex system that has been prepared for study in some specified manner. (A classic example would be one mole of helium occupying a one-liter enclosure in thermal equilibrium at a specified temperature.) There are two kinds of logical spectra that might conceivably arise in the quantal analysis of the system. One is related to the quantum states or *preparations* of the system; the other concerns the possible data that would emerge from subsequent *measurements* of observables of interest.

The logical spectrum associated with measurement of an observable \(A\) is obviously just the list of propositions of this form: "Measurement of \(A\) yields the datum \(a\)." In fact this logical spectrum of propositions concerning the results of \(A\)-measurements is indexed by the eigenvalue spectrum \({a_n}\) of the observable \(A\). It is customary to regard the projection operator onto the subspace belonging to \(a_n\) as the mathematical representative in quantum theory of that proposition of the above form which is indexed by \(a_n\). None of this is problematical.

The logical spectrum associated with the possible quantum states, on the other hand, is not so immediately identifiable. It is, however, the one which is of greatest interest in statistical physics, where the central problem is to make the best possible state assignment compatible with whatever meager physical information can be extracted from a description of the means employed in the laboratory to prepare the system of interest. We shall find that the problem of selecting such a logical spectrum of quantal state preparations leads almost at once into that thicket of quantal misunderstandings mentioned earlier.

There is of course an orthodox choice for the logical spectrum of quantum states. It has been entrenched for decades in all treatises on quantum statistics, and has been, as we shall see, willingly adopted also by the protagonists of the information-theoretic school. To construct that logical spectrum, let \({\psi_n}\) denote a *specific* complete orthonormal set of state vectors. The *standard logical spectrum* for states in quantum statistics consists of all propositions of this form: "The system is in the state \(\psi_n\)." Thus it is asserted that such a set of pure states is a mutually exclusive and exhaustive list of possibilities.

We maintain that this traditional choice of a logical spectrum for quantum statistics is a fundamental error, reflecting the imbuenment of a host of common misconstruals of the foundations of quantum mechanics.