Growth rates of the Marangoni instability in a layer of elastic liquid

Abstract The problem of thermo-capillary (Marangoni) convection in a layer of viscoelastic liquid is considered. The stability boundary for this problem has been previously calculated in various cases by a number of authors. Here attention is fixed on the magnitude of the growth rate in the parameter regime corresponding to instability. Two noteworthy features are pointed out. First, there are anomalously large values of the growth rate at or near the limiting special case of a Maxwell fluid. Second, the complex values of the growth rate (corresponding to overstability, or the onset of instability via oscillatory motion) coalesce into real (positive) values at moderately supercritical values of the Marangoni number, suggesting that overstability might be elusive to observation.

Key words Growth rate — Marangoni — elastic liquid

Introduction

When a horizontal layer of liquid is heated from below, convective instability will set in if the temperature gradient is large enough. The destabilising mechanism may originate in density gradients or in surface tension gradients, associated with the names Bénard and Marangoni respectively. Our purpose is to study the growth rate of this instability in the case when the liquid is viscoelastic.

This problem has a long history, of course, and there seems no reason to present a detailed survey here, as the particular aim of the present work can be adequately described without doing so. Instead the reader is referred to a recent paper by Lebon et al. (1994), not only for the historical sketch but also for the detailed derivation of the linearised disturbance equations which govern the stability problem. Here we shall use the same equations and boundary conditions and adopt the same notation for ease of reference.

Almost all the previous work has been concerned with the calculation of the stability boundary — that is, the set of values of the governing parameters which separates the stable and unstable regimes — and of various critical points on it. The present paper presents calculations of the growth rate of the disturbances in the unstable region. These calculations show several unexpected and interesting features; it will be easier to explain this in detail with the relevant equations to hand, and these are given in the next section.

Dimensionless linearised disturbance equations

We denote the vertical distance, measured from the solid boundary, by $x_3$, and the horizontal coordinates by $x_1$ and $x_2$. The disturbance vertical velocity and temperature are denoted $v'_3$ and $T'$ and are assumed to have the forms

$$
\begin{pmatrix}
v'_3 \\
T'
\end{pmatrix} = \begin{pmatrix}
W(x_3) \\
\theta(x_3)
\end{pmatrix} \exp \left[ i(k_1x_1 + k_2x_2) + \sigma t \right].
$$

(1)

The liquid is assumed to have a constitutive equation of state of the linearised Jeffreys type, with rate constants $L_1$ and $L_2$. The equations of momentum and energy reduce to
For a Newtonian liquid, it is known that this occurs when \( \text{Re}(a) = 0 \) but \( \text{Im}(a) \neq 0 \), termed “overstability” or “stationary transition”. The value of \( \text{Ma} \), say \( \text{Ma}_c \), at which this occurs can be calculated in terms of the other parameters, and \( \sigma \) is complex in general. As noted, attention until now has been concentrated on the stability boundary. For a Newtonian liquid, it is known that this is marked by \( \sigma = 0 \); and so when (say) \( \text{Ma} \) is increased, \( \sigma \) passes through zero via real values. This is termed “exchange of stabilities” or “stationary transition”. The value of \( \text{Ma} \), say \( \text{Ma}_c \), at which this occurs can be calculated in terms of the other parameters. It is of interest to find the least value of \( \text{Ma}_c \) as \( k \) varies, with the physical parameters held constant. This is because, if we imagine an experiment in which \( \text{Ma} \) is increased (by increasing the temperature gradient) quasi-statically, instability will set in at this least value and the wavenumber observed will be the corresponding value of \( k \).

For an elastic liquid it is found that the same transition occurs; this may be seen from the equations since if we put \( \sigma = 0 \) the terms in \( L_1 \) and \( L_2 \) drop out. However, there is a second possibility, that the transition is marked by \( \text{Re}(\sigma) = 0 \) but \( \text{Im}(\sigma) \neq 0 \), termed “overstability” or “oscillatory transition”. Here the growing disturbance is predicted to be oscillatory. It may happen that this transition occurs for smaller values of \( \text{Ma} \) than the transition through \( \sigma = 0 \), and in these cases we would expect to see oscillatory convection in the experiment outlined above.

No reports of observations of such oscillatory motions are known to the present author. This may cause us to question the robustness of the model described here, and is one reason why the calculations to be described here were undertaken. It turns out that the region of parameter space in which \( \sigma \) is complex is in many cases rather narrow. As \( \text{Ma} \) is increased from low values we find that \( \sigma \) is real and negative (even in the case of a Jeffreys liquid); then, as \( \text{Ma} \) is increased, \( \sigma \) bifurcates into a complex conjugate pair (with \( \text{Re}(\sigma) < 0 \)), crosses the imaginary axis at the stability boundary, and then coalesces again into a real positive value, at a value of \( \text{Ma} \) fairly close to the critical value. Of course only a small number of cases have been computed; but if these are typical, or at any rate not untypical, this could explain why overstability would be hard to observe, and would not commonly occur in what one might call real life, where the temperature gradient is established more or less accidentally.

Larson (1992) has remarked on the difficulty of producing the conditions for overstability for the Rayleigh problem; however in that case the difficulty stems from the fact that real materials will have material properties highly unlikely to be favourable.

Another motive for the present work relates to earlier investigations of the (real) growth rates in the unstable regime of the Taylor-Saffman problem (Wilson, 1990) and the Rayleigh-Taylor problem (Aitken and Wilson, 1993). Very large values of the growth rates were found for the Maxwell model, derived from the Jeffreys model by setting \( L_2 = 0 \), and similar results are found here. The values of \( \sigma \) are drastically reduced even for quite small values of \( L_2 \neq 0 \), as we shall later, and this suggests that the results for the Maxwell model are in a sense spurious. An explanation of these effects was offered by Aitken and Wilson (1993), who attributed them to the fact that the Maxwell liquid is instantaneously elastic. If this is accepted, then stability results of any kind for a Maxwell liquid would have to be regarded with some suspicion.

No attempt is made in this paper to give extensive numerical results, covering substantial ranges of the parameters; given the large number of parameters — four even in the much simplified model considered — that would hardly be possible or comprehensible. The aim is rather to draw attention to two qualitatively novel features and illustrate them by a sample. However, the author has searched in other regions of the parameter space, and the results given have not been specially selected — they are in fact quite typical.

### Results

We begin by limiting the number of parameters in the problem in order to simplify the results. First, it is reasonable to take \( Pr = \infty \) since \( Pr \) is generally large for polymeric liquids. We also put \( H = 0 \) and \( Ra = 0 \) simply in order to concentrate on one effect at a time. In fact, with this, the destabilising mechanism (the Marangoni effect) is confined to the boundary condition (6) via the parameter \( \text{Ma} \), and the equation which determines \( \sigma \) can be found analytically. This has two advantages for our purposes; it is possible to reduce the equation for \( \sigma \) to the point where the root structure can be grasped without any computation (though admittedly not without some effort), and further, the numerical information we require

\[
[(1 + L_2 \sigma) (D^2 - k^2)^2 + Pr^{-1} (1 + L_1 \sigma) \sigma (D^2 - k^2)]
\times W - k^2 Ra (1 + L_1 \sigma) \theta = 0
\]

\[
(D^2 - k^2 - \sigma) \theta + W = 0 ,
\]

where \( D = \frac{d}{dx_3} \) and \( k^2 = k_1^2 + k_2^2 \). The boundary conditions are

at \( x_3 = 0 \) \( \theta = W = D W = 0 \)

at \( x_3 = 1 \) \( D \theta + H \theta = 0 , W = 0 \)

\[ (1 + L_2 \sigma) D^2 W + k^2 Ma (1 + L_1 \sigma) \theta = 0 . \]