Geometric Significance of the Spinor Covariant Derivative

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The spinor covariant derivative through which the equations of quantum fields are generalized to include gravitational coupling has a direct and simple geometric significance. The formula for the difference of two spinor covariant derivatives taken in different order is derived geometrically; and the geometric proof of the covariant constancy of the spin-½ γ-matrices in curved space is given.

1. INTRODUCTION

The reader will not need to be reminded of the remarkable successes which the quantum theory of fields has met with in the description of phenomena on the atomic and subatomic scales, nor of the privileged position occupied by the general theory of relativity in the explanation of astrophysical and cosmological phenomena. There are perhaps no two branches of the body of theoretical physics whose mathematical and conceptual structures are so dissimilar, and yet the problem of the unification of the theory of gravitation with quantum theory has occupied the minds of some of the greatest physicists of our time.

At present, there are several known methods of describing the influence of gravitation on material phenomena, methods which are inevitably dualistic in that they all retain the conceptual distinction between the elementary notions of matter and gravitation. For instance, one such approach, pioneered by Schwinger, Feynman, and others, develops the theory of a massless, spin-2 field in the flat spacetime of the special theory of relativity. The predicted experimental consequences of the interaction of this field, whose quanta

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are gravitons, with other quantized fields found in nature are found to coincide, in all known cases, with those of the general theory of relativity. In particular, the lengths of the measuring rods and standard clocks that define a general coordinate mesh are also affected by gravitons, and the Minkowski spacetime that is the underlying spacetime arena of interacting graviton and matter fields itself becomes unobservable. In this way, the equivalence of the graviton quantum field theory with the general theory of relativity in the description of macroscopic phenomena stands established.

Apart from the theoretical and logical difficulties that are inherent in all such quantum field theories, this scheme for the quantization of the gravitational field does not lend itself very well to calculations of an astrophysical nature, in which one usually investigates phenomena on scales that are several orders of magnitude larger than the Planck length, and in which, therefore, the possible quantum behavior of gravitation is not manifested. It is generally more appropriate, in such situations, to quantize the matter field alone and to represent gravitation by the metrical field $g_{\mu\nu}(x)$, as in the general theory of relativity. The interaction between matter and gravitation is then described, not by adding interaction terms to the Lagrangian of the material system and its gravitational field, but by modifying the very equations followed by the matter field so as to make these equations conceptually meaningful in a curved spacetime background. The possibility of such a generalization of the equations of physics to a curved spacetime background is well known in the classical theory of gravitation, where it consists of the rule that all derivatives appearing in the equations are to be replaced by the corresponding covariant derivatives.

Two hypotheses concerning the equations of the matter field are introduced in this connection. The first hypothesis is that of general covariance of the equations under arbitrary nonsingular substitutions of the coordinates, a covariance that is well known from the theory of relativity. The second hypothesis is that the group of Lorentz transformations with respect to which the equations of the special theory of relativity are covariant can be generalized to curved space, and that the equations of the matter field are

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2 See Ref. 1, Chapter 10, Section 8.
3 There exists a variety of astrophysical systems for which models wherein the matter is quantized and gravity is continuous are thought to accord reasonably with physical reality, since the gravitational field is everywhere weak enough to be treated classically.
4 Due to the nonintegrability of direction in curved spacetime, however, such a rule does not, in general, give a unique generalization, and has no physical meaning unless there is added to it the assumption that the equations of physics can be put in so-called "canonical form," in which they contain no derivatives of order higher than the first (principle of compensation of gravitation over infinitesimally small regions). This should not be forgotten in what follows.