Algebraic Constraints on Hidden Variables

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In the contemporary discussion of hidden variable interpretations of quantum mechanics, much attention has been paid to the "no hidden variable" proof contained in an important paper of Kochen and Specker. It is a little noticed fact that Bell published a proof of the same result the preceding year, in his well-known 1966 article, where it is modestly described as a corollary to Gleason's theorem. We want to bring out the great simplicity of Bell's formulation of this result and to show how it can be extended in certain respects.

1. As a basis for comparison of Bell's(2) and Kochen and Specker's(5) algebraic no-hidden-variable proofs, we begin by outlining Kochen and Specker's strategy, omitting much of the technical detail. Kochen and Specker begin with two assumptions:

(i) Corresponding to each quantum state of a system, there is an underlying phase space \( \Omega \). Corresponding to each quantum mechanical observable \( A \) (self-adjoint operator on the Hilbert space of the system), there is a measurable function \( f_A : \Omega \to \mathbb{R} \) that assigns, for each point in the phase space \( \Omega \), a real number, which is to be thought of as the value of the observable for the point in the phase space.

(ii) For each observable \( A \) and each Borel function \( g \) we have \( g(f_A) = f_{g(A)} \), i.e., for each \( \omega \in \Omega \), \( g[f_A(\omega)] = f_{g(A)}(\omega) \).

Kochen and Specker then proceed by introducing the notions of a partial algebra of observables and a partial Boolean algebra of the projectors on the Hilbert space. Assumptions (i) and (ii) imply that there is an imbedding of the partial algebra of observables into a commutative algebra and an

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imbedding of the partial Boolean algebra of projectors into a Boolean algebra. This last in turn implies that there are homomorphisms from the partial Boolean algebra of projectors to the two-element Boolean algebra $Z_2$; indeed there is a homomorphism for each pair of distinct observables which separate the pair. Finally, they show that for each three-dimensional Hilbert space, there is a finite partial subalgebra of projectors that cannot be homomorphically mapped to $Z_2$, thereby showing that (i) and (ii) lead to a contradiction.

We want to describe the last step in the proof in a little more detail. Let $P_i, P_j, \ldots$ range over projectors onto one-dimensional subspaces $i, j, \ldots$ of a three-dimensional Hilbert space $H_3$. (We shall sometimes use these subscripts to refer to vectors spanning the subspaces.) A homomorphism from the partial Boolean algebra of these projectors to $Z_2$ assigns to each projector the value 1 or 0 (the maximal and minimal elements of $Z_2$); and if $i, j, k$ are pairwise orthogonal, then exactly one of $P_i, P_j, P_k$ gets mapped to 1 and the other two get mapped to 0. This can be pictured conveniently as a mapping of the points on the unit sphere in $H_3$ to 0 and 1 such that for each triple of orthogonal points on the sphere, one gets mapped to 1 and the other two get mapped to 0. Kochen and Specker use a straightforward geometrical argument to show that under these conditions, the angle subtended by points that get distinct values must be at least $\cos^{-1}1/2$. In other words, points that subtend a smaller angle must both receive the value 0 or both the value 1. The proof is then essentially done; since at least one point gets the value 1, all must get the value 1. But of each orthogonal triple, two are supposed to get the value 0. Furthermore, since all points within a cone of fixed small angle get assigned the same value, it is straightforward to show that there is a finite number of orthogonal triples for which the required assignment of 0's and 1's is impossible. Kochen and Specker’s continuation of the argument (Ref. 5, Lemma 2, pp. 68–69) merely counts the number of points needed, as we will illustrate below in the context of Bell’s work.

One is struck by the prominent role of the partial Boolean algebra of projectors in Kochen and Specker’s work. Judging by the frequent references in discussions of this work, it would appear that a formulation in terms of the partial Boolean algebra of projectors is generally thought to be the correct, or even the only way to understand this material. Using Bell’s presentation, we shall show shortly that this idea is mistaken. Indeed, we shall simplify the presentation of this material in two stages. In the first stage we can follow out an idea suggested by Kochen and Specker themselves and notice that each $\omega \in \Omega$ defines what we shall call a valuation function $v$; that is, a function that assigns an exact value (= real number) to each observable. Namely, for each observable $A$, let

$$v(A) = f_A(\omega)$$