AN APPLICATION OF THE NEKHOROSHEV THEOREM TO THE
RESTRICTED THREE-BODY PROBLEM

ALESSANDRA CELLETTI\textsuperscript{1} and LAURA FERRARA\textsuperscript{2}

\textsuperscript{1}Dipartimento di Matematica Pura e Applicata, Università di L'Aquila, Via Vetoio, I-67010 Coppito (L'Aquila), Italy (e-mail: Celletti@axscag.aquila.infn.it)
\textsuperscript{2}Corso di Laurea in Matematica, Università di L'Aquila, Via Vetoio, I-67010 Coppito (L'Aquila), Italy (e-mail: Ferrara@axscag.aquila.infn.it)

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Abstract. We studied the stability of the restricted circular three-body problem. We introduced a model Hamiltonian in action-angle Delaunay variables, which is nearly-integrable with the perturbing parameter representing the mass ratio of the primaries. We performed a normal form reduction to remove the perturbation in the initial Hamiltonian to higher orders in the perturbing parameter. Next we applied a result on the Nekhoroshev theorem proved by Pöschel [13] to obtain the confinement in phase space of the action variables (related to the elliptic elements of the minor body) for an exponentially long time. As a concrete application, we selected the Sun–Ceres–Jupiter case, obtaining (after the proper normal form reduction) a stability result for a time comparable to the age of the solar system (i.e., $4.9 \cdot 10^9$ years) and for a mass ratio of the primaries less or equal than $10^{-6}$.

Key words: Stability, Nekhoroshev theorem, three-body problem

1. Introduction

One of the most powerful results concerning the stability of nearly-integrable Hamiltonian systems is provided by the Nekhoroshev theorem [11]. The theorem, in its original version, was formulated for a wide class of Hamiltonians and, under suitable assumptions, it guarantees the stability of the action variables over exponentially long times.

In recent years, different proofs of the theorem were proposed [1, 2, 9, 13]. In this paper we apply a result due to Pöschel [13] to a problem of Celestial Mechanics. More specifically, we investigate the stability of the restricted circular three-body problem, which has been studied through perturbation theory by several authors [4, 5, 8, 10, 12]. Let $P_1, P_2$ be two bodies (with masses $m_1, m_2$) moving in the gravitational field of a central body $P_0$ (with mass $m_0$). We assume that $m_1 \ll m_2$, so that the dynamics of $P_2$ is not affected by $P_1$ and we impose that $P_2$ moves on a circular Keplerian orbit. Finally, we assume that the motion of $P_1$ takes place on the same plane of the orbit of the primaries. The Hamiltonian function associated to this model can be expressed in terms of suitable Delaunay action-angle variables $(L, G, l, g)$, which are closely related to the elliptic elements of $P_1$. In particular, $L = \sqrt{\mu a}$ and $G = \sqrt{\mu a(1 - e^2)}$, where $\mu \equiv m_0 + m_1$ and $a, e$ are the semimajor axis and the eccentricity of the Keplerian orbit on which $P_1$
would move in absence of $P_2$. With an appropriate choice of the units of measure, one gets a Hamiltonian of the form:

$$H(L, G, l, g) = \frac{1}{2L^2} - G + \varepsilon R(L, G, l, g), \quad (1)$$

where $\varepsilon \equiv m_2/m_0 (\varepsilon < 1)$ and $R(L, G, l, g)$ is the perturbing function, representing the interaction between $P_1$ and $P_2$. We further reduce this model retaining only the most significant terms in the Fourier series expansion of $R(L, G, l, g)$ (standard computations show that the Fourier coefficients decay as powers of the orbital eccentricity, which we assume much less than one).

We have learned in previous works (see, e.g., [3]) that before applying the stability theorem, it is usually convenient to perform a canonical change of variables, $(L, G, l, g) \rightarrow (L', G', l', g')$, in order to reduce (1) to the form

$$H^{(n)}(L', G', l', g') = h^{(n)}(L', G') + \varepsilon^{n+1} f^{(n)}(L', G', l', g'), \quad (2)$$

for a given integer $n > 0$. Notice that, in general, the size of the perturbation in (2) is smaller than in the original Hamiltonian function. Once obtained the explicit expression of (2), we apply the Nekhoroshev theorem provided in [13].

As a specific application, we consider the Sun–Ceres–Jupiter system with an Hamiltonian function containing 17 Fourier coefficients. Using a computer we construct the normal form (2) up to the order $n = 4$. Our result can be summarized as follows.

Let $(L_0, G_0) = (0.72925656, 0.72710370)$ and $(\tilde{L}_0, \tilde{G}_0) = (0, 0)$; for a mass ratio $\varepsilon \leq \varepsilon^* = 10^{-6}$, the action variables satisfy

$$\|L(t) - \tilde{L}_0\| \leq 1.6051 \cdot 10^{-7}, \|G(t) - \tilde{G}_0\| \leq 1.5492 \cdot 10^{-7},$$

$$\forall |t| \leq T = 4.9277 \cdot 10^9 \text{ years.}$$

The initial conditions $(\tilde{L}_0, \tilde{G}_0)$ are close (up to $10^{-4}$) to the exact location of Ceres. Therefore the domain of variation of the action variables does not include the position of Ceres; unfortunately, one obtains worst results taking initial conditions closer to Ceres, due to the presence and accumulation of small divisors. Moreover, we have not been able to prove the stability for a mass ratio $\varepsilon^*$ consistent with the actual observations (for Jupiter–Sun the mass ratio amounts to $\varepsilon^* = 10^{-3}$).

However, the above value is still a remarkable result in the context of three-body stability. The same model has been recently studied in [4], using the Kolmogorov–Arnold–Moser theorem on the existence of invariant surfaces.

This paper is organized as follows. In Section 2 we recall the Nekhoroshev theorem provided by Pöschel [13]. In Section 3 we introduce our three-body model Hamiltonian. The normal form theory is developed in Section 4. The results are presented in Section 5.