Relativistic Classical Mechanics and Canonical Formalism

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The analysis of interacting relativistic many-particle systems provides a theoretical basis for further work in many diverse fields of physics. After a discussion of the nonrelativistic N-particle systems we describe two approaches for obtaining the canonical equations of the corresponding relativistic forms. A further aspect of our approach is the consideration of the constants of the motion.

1. INTRODUCTION

Interacting classical relativistic systems have recently been constructed by several authors. \(^1\) \(^-\) \(^9\) Within these approaches the so-called no-interaction theorem \(^1\) \(^0\) \(^-\) \(^1\) \(^1\) is overcome.

In the present paper we describe two approaches for classical relativistic two-body problems. We study a classical system of \(N\) spinless particles. The approach is motivated by the modern formulation of nonrelativistic classical mechanics using symplectic manifolds. The formulation of nonrelativistic classical mechanics in the modern language of symplectic geometry has been fully developed. \(^1\) \(^2\) \(^3\)

Recently Horwitz and Piron, \(^1\) Piron and Reuse, \(^2\) and Reuse \(^3\) described a canonical formalism for the relativistic classical mechanics of many particles. Within the given approach the authors discussed the evolution equations for a charged particle in an electromagnetic field and the relativistic two-body problem. In the present paper we describe how the evolution equations given by these authors can be obtained from a different
approach. We compare both approaches. For the sake of completeness we also consider nonrelativistic classical mechanics. Moreover, the constants of the motion are given. We derive a theorem\textsuperscript{(14)} for obtaining constants of the motion which includes the nonrelativistic and relativistic classical mechanics. Another approach, namely the so-called constraint Hamiltonian formulation\textsuperscript{(6-8)} of relativistic point particle dynamics, is only briefly considered in the present paper.

Throughout we deal with differential forms and vector fields and with the concept of the Lie derivative of a differential form (or a vector field) with respect to a vector field. The invariance conditions are formulated with the help of the Lie derivative.\textsuperscript{(15)}

2. NONRELATIVISTIC CLASSICAL MECHANICS

In this section we briefly describe the approaches for obtaining the equations of motions in nonrelativistic classical mechanics. In Section 3 we extend the approaches to the relativistic case. In the first approach the starting point is the action one-form

\[
\alpha = \sum_{i=1}^{N} \sum_{j=1}^{3} p_{ij} dq_{ij} - H(p_{11}, \ldots, q_{N3}) \, dt
\]  

Given a closed curve \( C \) in \( \mathcal{F} = \{p_{11}, \ldots, p_{N3}, q_{11}, \ldots, q_{N3}, t\} \), the requirement that the integral

\[
\oint_C \alpha
\]

is invariant for any continuous deformation of \( C \) obtained by arbitrary displacement of its points along the trajectories leads to the canonical equations

\[
dp_{ij}/dt = -\partial H/\partial q_{ij}, \quad dq_{ij}/dt = \partial H/\partial p_{ij}
\]

In other words the equations of motion are determined by \( Z \lrcorner \, d\alpha = 0 \) (\( \lrcorner \) denotes the contraction of a differential form with a vector field) where \( \alpha \) is given by Eq. (1) and the vector field \( Z \) takes the form

\[
Z = \sum_{i=1}^{N} \sum_{j=1}^{3} \left( V_{ij} \frac{\partial}{\partial p_{ij}} + W_{ij} \frac{\partial}{\partial q_{ij}} \right) + \frac{\partial}{\partial t}
\]