Variations on a Wignerian Theme

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The Wigner distribution and its equation of motion in the scalar potential ease are arrived at in an unusual way. This in turn suggests (a) a departure from the standard Wigner distribution treatment for a charged particle in a magnetic field and (b) a new approach to quantization of nonconservative systems. Suggestion (a) is found to be, like the standard treatment, in agreement with Schrödinger's equation but, unlike it, also satisfies local classical-type conservation laws and employs a distribution which is gauge-invariant rather than merely gauge-covariant. Suggestion (b) gives a clear result only in the case of resistance proportional to velocity, when it agrees with the Schrödinger-Langevin equation; for other dissipative systems a fresh assumption is required, and a proposal in that direction is put forward.

1. INTRODUCTION AND SUMMARY

Our theme is the Wigner\(^{(1)}\) distribution over position \(x\) and momentum \(p\) which, for a single particle in three dimensions with wave function \(\psi\), may be written

\[
\mathcal{F}(x, p) = \left(\frac{1}{2\pi}\right)^3 \int e^{i p \cdot s} \psi^*(x - \frac{1}{2} \hbar s) \psi(x + \frac{1}{2} \hbar s) \, ds
\]  

integration being over the whole space of the auxiliary vector variable \(s\). The expected value of \(\phi\) when \(\phi = F(x)\) or \(\phi = G(p)\) is then equal to the integral of \(\mathcal{F}\) over the whole phase space—a result which, with proper interpretation, can be extended to more general \(\phi(x, p)\).

The presentation of this theme in Sections 2–4, is unusual in two ways. The distribution is reached by an imaginative leap from the notions of

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probability density and probability current density; and its evolution in time is seen as departing from Liouville's equation, not on account of further quantum mechanical considerations, but through the maintenance of (1) by local redistribution processes satisfying what may be described as classical principles of local conservation of material and momentum. If the equations of motion of \( f \) have not been viewed thus hitherto, it may well be because, as shown in Section 5, the method seems to fail for a charged particle in a magnetic field.

This leads to our first variation. In the orthodox Wigner distribution (1), \( p \) signifies momentum canonically conjugate to \( x \) in all cases. But the probability current density approach to the distribution suggests that (1) should be used only in the case of a scalar potential; in the presence of a vector potential it suggests instead the distribution, discussed in Section 6, over position and mechanical momentum (\( = \text{mass} \times \text{velocity} \)). Moreover, the classical-type treatment of the equations of motion of this distribution given in Section 7 turns out to be in agreement with Schrödinger's equation.

The treatment may be called Newtonian in the sense that forces enter directly into the equations of motion and not via a Lagrangian or Hamiltonian. In our second variation, in Section 8, it is used to deal with resistance proportional to velocity, giving a result in agreement with the Schrödinger–Langevin equation. The combination of scalar potential, vector potential, and resistance proportional to velocity exhausts the immediate scope of the treatment, because of problems of integrability of the equations of motion in other cases. If the Schrödinger–Langevin equation is indeed the definitive expression in the Schrödinger picture appropriate to resistance proportional to velocity, as we would maintain, it could be argued that what has been achieved is no more than a reformulation of known work and is thus perhaps of scholastic (e.g., didactic, philosophical, or mathematical) rather than physical interest.

However, a more radical look, in Section 9, at the way the difficulty with the magnetic field was overcome suggests a further, final, variation in order to surmount integrability problems associated with forces given by more general functions of position and velocity. This suggestion, discussed in Section 10, and perhaps applicable to radiation damping, has a new feature which may raise more problems than it solves. Whether this counts in its favor or against it remains to be seen.

### 2. THE QUANTUM CONSTRAINT—SIMPLE VERSION

In seeking, in the one-dimensional case, a distribution \( f(x, p, t) \) in phase space to be associated with the wave function \( \psi \) of a particle of mass \( m \) at