Coulomb Gas Representation of the SU(2) WZW Correlators at Higher Genera

Krzysztof Gawedzki
CNRS, IHES, 35 route de Chartres, 91440 Bures-sur-Yvette, France

(Received: 5 July 1994)

Abstract. We extend a previous analysis to the case with insertion points. The result allows us to express the correlation functions of the SU(2) WZW conformal field theory on Riemann surfaces of genus > 1 by finite-dimensional integrals.


1. Introduction

The present Letter completes the work [1] where we computed the scalar product of the SU(2) Chern–Simons states on a Riemann surface of genus > 1 in the absence of Wilson lines. Here, we treat the case with Wilson lines $c_i$, in representations of spin $j_i$, cutting the Riemann surface at marked points. Let us briefly list the basic notations, definitions, and relations, referring to [1] for a more extensive introduction. Below:

- $\Sigma$ denotes the Riemann surface (of genus $\gamma > 1$) with distinct marked points $\xi_l, l = 1, \ldots, L$,
- $\mathcal{A}^{01}$ stands for the space of smooth $\text{sl}(2, \mathbb{C})$-valued 0, 1-forms $A^{01}$ on $\Sigma$,
- $\mathcal{G}^C$ is the group of complex gauge transformations given by smooth maps $h: \mathcal{A}^{01} \rightarrow \text{SL}(2, \mathbb{C})$ which act by $A^{01} \mapsto hA^{01}h^{-1} + h\partial h^{-1}$ on $\mathcal{A}^{01}$,
- $S(h, A^{01})$ denotes the WZW model action in the presence of the gauge field,
- $k = 1, 2, \ldots$ is the level of the theory,
- $V_j$ stands for the space of spin $j$ representation with $g \in \text{SL}(2, \mathbb{C})$ acting on it by linear automorphism $g_j$,
- $\Psi: \mathcal{A}^{01} \rightarrow \bigotimes_i V_{j_i}$ is a CS state if it is holomorphic and if $^h \Psi = \Psi$, where

$$ ( ^h \Psi)(A^{01}) = \exp[-kS(h, A^{01})] \otimes_i h(\xi_i)_{j_i} \Psi(h^{-1} A^{01}),$$

(1.1)

$W_k((\xi_i), (j_i))$ denotes the (finite-dimensional) space of the CS states.

The scalar product of the CS states is formally given by the integral

$$ \| \Psi \|^2 = \int |\Psi(A^{01})|_V^{2j_i} \exp \left[ \frac{ik}{2\pi} \int_\Sigma \text{tr} A^{10} \wedge A^{01} \right] DA. \quad (1.2)$$
The aim of this Letter is to compute the functional integral (1.2) over the \( su(2) \) gauge fields \( A = A^{10} + A^{01} \) with \( A^{10} = -(A^{01})^\dagger \) by reducing it to an explicit finite-dimensional integral. Such a reduction allows us to express the correlation functions in the external gauge field of the WZW model of conformal field theory,

\[
\Gamma((\xi_l), (j_l), A) = \int \bigotimes_l g(\xi_l)_{j_l} \exp[-kS(g, A)]Dg,
\]

by finite-dimensional integrals. According to [2],

\[
\Gamma((\xi_l), (j_l), A) = \sum_{r,r'} H^{rr'}(A^{01}) \otimes \overline{\Psi_r(A^{01})} \exp \left[ \frac{ik}{2\pi} \int \text{tr} A^{10} \wedge A^{01} \right]
\]

for any basis \((\Psi_r)\) of \( W_k((\xi_l), (j_l))\) with matrix \((H^{rr'})\) inverting the matrix \(((\Psi_r, \Psi_r))\) of scalar products of \( \Psi_r\)'s. Hence,

\[
\Gamma((\xi_l), (j_l), A) = \sum_{r,r'} H^{rr'}(A^{01}) \otimes \overline{\Psi_r(A^{01})} \exp \left[ \frac{ik}{2\pi} \int \text{tr} A^{10} \wedge A^{01} \right] \times \]

\[
\times \int z^r \overline{z}^{r'} \exp \left[ -\sum_{s,s'} \overline{z}^s(\Psi_s, \Psi_{s'}) z^{s'} \right] \prod_t d^2 z_t \times \]

\[
\times \int \exp \left[ -\sum_{s,s'} \overline{z}^s(\Psi_s, \Psi_{s'}) z^{s'} \right] \prod_t d^2 z_t \quad (1.5)
\]

into which one should substitute the finite-dimensional integral expressions for the scalar products \( (\Psi_s, \Psi_{s'})\).

The functional integral in Equation (1.2) is calculated by a change of variables \( A^{01} \rightarrow (h, n) \),

\[
A^{01} = h^{-1} A^{01}(n), \quad (1.6)
\]

where \( h \in G^C\) and \( n \) parametrizes a slice \( \{ A^{01}(n) \} \) (of complex dimension \( 3(\gamma - 1) \equiv N \) in \( G^{01} \) which cuts a generic orbit of \( G^C \) a finite number, say \( \nu \), of times. Upon the change of variables, Equation (1.2) becomes

\[
\| \Psi \|^2 = \frac{1}{\nu} \int \bigotimes_l h(\xi_l)_{j_l}^{-1} \Psi(A^{01}(n)) \bigotimes_{\nu j_l} \]

\[
\times \exp \left[ -\frac{ik}{2\pi} \int_{\Sigma} \text{tr}(A^{01}(n))^* \wedge A^{01}(n) \right] \exp[(k + 4)S(hh^\dagger, A(n))] \times \]

\[
\times \det(\hat{D}_n ^\dagger D_n) \det(\Omega(1, n))^{-1} \left| \det \left( \int_{\Sigma} \omega(\theta(n) \wedge \frac{\partial A^{01}(n)}{\partial n_\alpha}) \right)^2 D(hh^\dagger) \prod x d^2 n_\alpha \right|^2 \quad (1.7)
\]