CATEGORICITY OF FINITELY GENERATED ALGEBRAIC SYSTEMS IN HF-LOGIC

V. Ya. Belyaev, E. E. Lyutikova, and V. N. Remeslennikov

The article inquires into the problem of whether or not finitely generated algebraic systems are categorical in HF-logic. A system \( A \) is called HF-categorical if, for any system \( B \), \( A \cong_{HF} B \) implies \( A \cong B \). The main result is the construction of models, an associative ring in particular, which are finitely generated but not categorical in HF-logic. Finitely generated systems with arithmetic diagrams are all HF-categorical. Starting with hyperarithmetic level, however, there are systems which are HF-categorical and which are not.

INTRODUCTION

Since the ordinary first-order logic often fails to describe many of the natural algebraic properties, it seems reasonable to use stronger logics to meet this purpose. It is desirable that such a logic be equipped with adequate tools for studying properties of algebraic objects, contain the first-order logic, and its formulas satisfy some reasonable requirements of effectiveness.

Of major concern in this respect is the weak second-order logic, also known as the logic of hereditarily finite types, or HF-logic. This logic is natural in an algebraic setting since it allows us to directly deal with natural numbers, finite sets, finite sequences, etc. In HF-logic, almost all natural properties of an algebraic system of finite signature are expressible. These include, for example, the properties “to be finitely generated,” “to belong to a recursive variety,” “to be a free algebra of finite rank in a variety of this kind,” etc.

Of interest is the problem of describing algebraic systems that are categorical in HF-logic. Recall that a system \( A \) is called HF-categorical if, for any system \( B \), \( A \cong_{HF} B \) implies \( A \cong B \).

In this paper, we deal with the case where \( A \) is finitely generated. The survey presented in Sec. 2 shows that virtually all explicitly presented systems are HF-categorical. These include, in particular, all systems generated by the empty set (i.e., by constants in the signature), all systems for which the word problem is arithmetical or merely implicitly definable in arithmetic, etc.

Our main result is the construction of models (an associative ring in particular) which are finitely generated but not categorical in HF-logic. Finitely generated systems with arithmetical diagrams are all HF-categorical. Starting with hyperarithmetic level, however, there exist systems which are determined, up to an isomorphism, by their HF-theories and which are not.
1. DEFINING THE HF-LOGIC

Let $\mathcal{A} = \langle A; \sigma \rangle$ be an algebraic system of a finite signature. Consider the superstructure over $\mathcal{A}$, which is denoted by $\text{HF}(\mathcal{A})$ and is defined in the following manner. For an arbitrary set $M$, $F(M)$ stands for the set of all finite subsets of $M$. Define the set $\text{HF}(\mathcal{A})$ by induction as follows:

$$A_0 = F(A), \quad A_{n+1} = F(A \cup A_n), \quad \text{HF}(\mathcal{A}) = \bigcup_{n=0}^{\infty} A_n.$$ 

Then

$$\text{HF}(\mathcal{A}) = \langle A \cup \text{HF}(\mathcal{A}), \sigma, \in \rangle,$$

where $\in$ is the membership relation on $A \cup \text{HF}(\mathcal{A})$. This is an ordinary algebraic system of the signature $\sigma^* = \sigma \cup \{\in(2)\}$. To be specific, we assume that all predicates from $\sigma$ are false at the elements of $\text{HF}(\mathcal{A})$, and that all operations on $\text{HF}(\mathcal{A})$ take the value $\emptyset \in \text{HF}(\mathcal{A})$. Elements of $A$ are called urelements. No element of $A \cup \text{HF}(\mathcal{A})$ is assumed to belong to an element of $A$. Formulas of the language of HF-logic ($L_{HF}$) in the signature $\sigma$ are formulas in the first-order language with equality of the signature $\sigma^*$. Thus, studying the system $\mathcal{A}$ in HF-logic is equivalent to treating the system $\text{HF}(\mathcal{A})$ in the ordinary first-order logic.

Definition 1.1. Algebraic systems $\mathcal{A}$ and $\mathcal{B}$ are called HF-equivalent, and we write $\mathcal{A} \equiv_{HF} \mathcal{B}$, if each sentence of HF-logic that is true in $\mathcal{A}$ is also true in $\mathcal{B}$, and vice versa.

The superstructure $\text{HF}(\mathcal{A})$ is in fact the minimal admissible set with urelements from $A$. Therefore, many logical results on admissible sets (see, e.g., [1]) are directly adaptable to the setting of HF-logic. In particular, the set of natural numbers $\mathbb{N}$ ($0 \rightarrow \emptyset$, $n+1 \rightarrow n \cup \{n\}$) and all recursively enumerable relations on this set are interpreted in $\text{HF}(\mathcal{A})$ by $\Sigma$-formulas.

In [2], a fragment of the language $L_{\omega_1\omega}$, denoted by $L_{AR}$, was described. Its expressive power is equivalent to that of HF-logic, and for $L_{AR}$, we do not need to extend the signature or to build a superstructure. Recall the definition of that language. For $\varphi \in L_{\omega_1\omega}$ an arbitrary formula, we define the following concepts: $\varphi \in L_{AR}$, the rank $r(\varphi)$ of $\varphi$ is not greater than $n$, where $n$ is a natural number, and $\nu(\varphi)$ is the number of $\varphi$.

(a) $L_{AR}$ contains all atomic first-order formulas in the signature $\sigma$; for the $\varphi$, we assume that $r(\varphi) \leq n$ for each $n$, and $\nu(\varphi)$ is the Gödel number of $\varphi$.

(b) Let $\varphi$ be one of the following: $\varphi_1 & \varphi_2$, $\varphi_1 \lor \varphi_2$, $\varphi_1 \rightarrow \varphi_2$, $\varphi_1 \leftrightarrow \varphi_2$, or $\neg \varphi_1$. Assume that $\varphi_1 \in L_{AR}$, $\varphi_2 \in L_{AR}$, $\nu(\varphi_1)$, and $\nu(\varphi_2)$ are defined, and for each $n$, we know whether or not the statements $r(\varphi_1) \leq n$ and $r(\varphi_2) \leq n$ are true. Then

$$\varphi \in L_{AR};$$

$$r(\varphi) \leq n \iff r(\varphi_1) \leq n \text{ and } r(\varphi_2) \leq n;$$

$\nu(\varphi)$ is effectively computed from $\nu(\varphi_1)$ and $\nu(\varphi_2)$.

(c) Let $\varphi \in L_{AR}$; then $\forall x \varphi(x) \in L_{AR}$ and $\exists x \varphi(x) \in L_{AR};$

$$r(\forall x \varphi(x)) \leq \begin{cases} n & \text{if } r(\varphi) \leq n \text{ and } \varphi = \forall y \psi; \\ n + 1 & \text{if } r(\varphi) \leq n \text{ and } \varphi \text{ is not prefixed with } \forall; \end{cases}$$

$$r(\exists x \varphi(x)) \leq \begin{cases} n & \text{if } r(\varphi) \leq n \text{ and } \varphi = \exists y \psi; \\ n + 1 & \text{if } r(\varphi) \leq n \text{ and } \varphi \text{ is not prefixed with } \exists. \end{cases}$$