The Inverse Scattering Matrix for the Schrödinger Equation when the Potential $q(x) \in L^1$ with a Singular Term

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Abstract. When the potential $q(x) \in L^1$ with a singular term, the continuities of the scattering matrix of the Schrödinger equation are investigated. By means of the transformation approach, we arrive at the conclusion that the scattering matrix $S(k)$ of such a potential is continuous for the whole $k$, $-\infty < k < \infty$.

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1. Introduction

The self-adjoint Schrödinger operator $H$ in $L^2(-\infty, \infty)$ is defined as

$$H \equiv \text{Friedrichs extension of } \left(-\frac{d^2}{dx^2} + q(x)\right) \upharpoonright C_0^\infty(-\infty, \infty),$$

where

$$q(x) \in L^1,$$

$$L^1_n = \left\{p(x) \text{ measurable, } \int_{-\infty}^{\infty} (1 + |x|^n) |p(x)| dx < \infty \right\}.$$

Then $H$ has both a pure absolutely continuous spectrum $[0, \infty]$ and a finite number of bound states below continuum $[1, 2]$. Let $f_j(x, k), j = 1, 2$ be the solutions of $Hf_j = k^2f_j, j = 1, 2$ with

$$f_1(x, k) \sim e^{ikx}, \quad \text{as } x \to \infty,$$

$$f_2(x, k) \sim e^{-ikx}, \quad \text{as } x \to \infty,$$

i.e. Jost functions. Then

$$T_2(k)f_1(x, k) \sim e^{ikx} + R_2(k) e^{-ikx}, \quad \text{as } x \to -\infty,$$

$$T_1(k)f_2(x, k) \sim e^{-ikx} + R_1(k) e^{ikx}, \quad \text{as } x \to \infty.$$
The solution \( T_2(k)f_1(x, k) \) represents a plane wave \( e^{ikx} \) sent in from \( -\infty \), transmitting \( T_2(k) e^{ikx} \) to \( +\infty \) and reflecting \( R_2(k) e^{-ikx} \) to \( -\infty \), while \( T_1(k)f_2(x, k) \) represents the scattering from \( +\infty \).

The matrix

\[
S(k) = \begin{bmatrix} T_1(k) & R_2(k) \\ R_1(k) & T_2(k) \end{bmatrix}, \quad k \in \mathbb{R}/\{0\},
\]

is called the scattering matrix (S-matrix) for \( q(x) \).

As is well known, the study of the properties of the scattering matrix \( S(k) \) is a very fundamental problem. Firstly, Faddeev [3, 4] proved that the matrix \( S(k) \) is continuous at \( k = 0 \), when \( q(x) \in L^1 \). However, Chadum and Sabater [5] point out the error of the proof in [4], and claim that when \( q(x) \in L^1 \), \( S(k) \) is not continuous at \( k = 0 \). Later, Deift and Trubowitz [6] reconsidered this question and proved that when \( q(x) \in L^1 \), \( S(k) \) is continuous at \( k = 0 \), while, when \( q(x) \in L^1 \), the continuous question of \( S(k) \) at \( k = 0 \) is still an open problem. In the scattering theories, e.g., nuclear physics, the nature of the function possesses a physical significance that is a more general function \( q(x) = O(x^{-1}) \). Thus, the research of this question has such a physical background. In this Letter, we prove that \( S(k) \) is continuous at \( k = 0 \) in Section 2, when \( q(x) \in L^1 \). Following, in Section 3, moreover, when the potential \( q(x) \in L^1 \) with a singular term, the continuity of \( S(k) \) at \( k = 0 \) is proven.

2. The Continuity of the S-Matrix when \( q(x) \in L^1 \) without a Singular Term

Using the approach of translating into the general equation, we can readily justify that the Jost functions \( f_1(x, k), f_2(x, k) \) exist and for \( k \neq 0, f_j(x, k), f_j(x, -k), j = 1, 2 \), are two linear independent solutions, when \( q(x) \in L^1 \). Based on the above statement, we can derive that for real \( k \neq 0 \), there is a unique set of functions \( T_1(k), T_2(k), R_1(k), R_2(k) \), satisfying (1.4) and (1.5). Thus, for \( k \neq 0 \), the scattering matrix \( S(k) \) exists and

\[
[T_1(k)]^{-1} = [f_1(x, k), f_2(x, k)]/2ik = [T_2(k)]^{-1},
\]

\[
\frac{R_1(k)}{T_1(k)} = \frac{[f_2(x, k), f_1(x, -k)]}{2ik},
\]

\[
\frac{R_2(k)}{T_2(k)} = \frac{[f_2(x, -k), f_1(x, k)]}{2ik},
\]

and

\[
\frac{R_2(k)}{T_1(k)} = \frac{1}{2ik} \int_{-\infty}^{\infty} e^{2ikt} q(t) m_1(t, k) \, dt,
\]

\[
\frac{1}{T_1(k)} = 1 - \frac{1}{2ik} \int_{-\infty}^{\infty} e^{-2ikt} q(t) m_1(t, k) \, dt,
\]

\[
\frac{R_1(k)}{T_2(k)} = \frac{1}{2ik} \int_{-\infty}^{\infty} e^{-2ikt} q(t) m_2(t, k) \, dt,
\]

\[
\frac{R_2(k)}{T_2(k)} = \frac{1}{2ik} \int_{-\infty}^{\infty} e^{2ikt} q(t) m_2(t, k) \, dt,
\]

where

\[
m_1(t, k) = \frac{1}{2ik} \int_{-\infty}^{\infty} e^{ik(t-s)} q(s) m_1(s, k) \, ds,
\]

\[
m_2(t, k) = \frac{1}{2ik} \int_{-\infty}^{\infty} e^{-ikt} q(t-s) m_2(s, k) \, ds.
\]