Grading of Spinor Bundles and Gravitating Matter in Noncommutative Geometry

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Abstract. The gravitating matter is studied within the framework of noncommutative geometry. The noncommutative Einstein–Hilbert action on the product of a four-dimensional manifold with discrete space gives models of matter fields coupled to the standard Einstein gravity. The matter multiplet is encoded in the Dirac operator which yields a representation of the algebra of universal forms. The general form of the Dirac operator depends on a choice of the grading of the corresponding spinor bundle. A choice is given, which leads to the nonlinear vector σ-model coupled to the Einstein gravity.


1. Introduction

Geometrical methods in theoretical physics are of generally recognized relevance for model building. Unravelling the geometrical structure in a theory usually leads to new insights and results not achieved previously. Indeed, the dynamics of non-Abelian gauge theories or of string theory is closely connected with their geometrical structure [1, 2]. Apart from the use of standard geometrical methods, the developments of the last decade in mathematical physics have led to applications of ideas of the so-called noncommutative geometry proposed by A. Connes. As an example, we can take Witten’s open string field theory [3] or the formulation of the standard model using ideas of noncommutative geometry [4–6]. The latter application was developed by Connes himself and, among other things, it gave geometrical meaning to the Higgs field. Connes’ approach is, in some sense, a small deviation from standard commutative geometry, because the notion of a point retains its sense. Noncommutativity enters when metric properties are defined.
Recently, A. H. Chamseddine, G. Felder, and J. Fröhlich [7] have introduced Einstein–Hilbert gravity within the noncommutative geometry framework. They considered spacetime as the product of a standard four-dimensional manifold with a Kaluza–Klein-like internal space consisting of two points. Metric aspects are encoded in the notion of the Dirac operator acting on the direct sum \( H \) of two copies of the space of spinor fields. An important role in such a construction is played by the grading on the space \( H \). The Dirac operator is required to be odd, hence its general form clearly depends on a choice of the grading.

In this Letter, we wish to elaborate on this point. We pick up a different grading than the one discussed in [7] and we consider the most general Dirac operator to be consistent with this grading. While in the Chamseddine, Felder, and Fröhlich approach, the general Dirac operator depends on a standard (commutative) vierbein and two scalar fields, in our case the scalar fields are replaced by one vector field. Using the noncommutative Einstein–Hilbert action and imposing the zero (noncommutative) torsion condition, we get a nonlinear vector \( \sigma \)-model coupled to the standard gravity. The resulting Lagrangian has a quite unexpected form and it indicates the richness of the types of models obtainable from the noncommutative geometry approach.

2. Elements of Noncommutative Geometry

Let us consider a smooth compact four-dimensional spin manifold \( Y \) with a fixed spin structure and a real algebra \( \mathcal{C}^\infty(Y) \) of smooth functions on it. The commutative algebra \( A \) is defined as the algebra of diagonal matrices

\[
 f = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}
\]

with values in \( \mathcal{C}^\infty(Y) \). Elements of \( A \) can clearly be considered as smooth functions on the product \( Y \times Z_2 \).

Let \( S \) denote the associated spinor bundle on \( Y \). The Riemannian metric on \( Y \) determines the corresponding Clifford bundle \( \mathcal{C}(T^* Y) \). Its complexification coincides with the bundle \( \text{End}(S) \), sections of \( \mathcal{C}(T^* Y) \) are called 'real sections of \( \text{End}(S) \)'. The group \( Z_2 \) acts by permutation on the bundle \( \tilde{S} = S \oplus S \), equivariant linear operators on sections of \( \tilde{S} \) are those commuting with the action.

Then we introduce the Dirac K-cycle \((H, D, \Gamma)\), where

(i) the Hilbert space \( H \) is the sum of two copies of the Hilbert space of \( L_2 \)-sections of the spinor bundle \( S \),

(ii) the grading \( \Gamma \) is defined by

\[
 \Gamma = \begin{pmatrix} \gamma_5 & 0 \\ 0 & \gamma_5 \end{pmatrix},
\]