A Discussion of the Pseudo-Weyl-Type Metrics

JOSEPH F. TOMBRELLO and JOHN H. YOUNG

Department of Physics, The University of Alabama in Birmingham, Birmingham, Alabama 35294

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Abstract

We discuss the "pseudo-Weyl" vacuum field for which the two metric functions, $\psi$ and $\gamma$, depend on the three coordinates $\rho, z$, and $t$. It is shown that no solutions exist that depend on all three coordinates. Consideration is given to the time-dependent metric of Einstein and Rosen and the same (null) result is shown to hold for that case. Thus, the most general solutions to the Weyl-type metric appear to be those already found by Weyl and by Einstein and Rosen.

The nonlinearity of the system of coupled second-order partial differential equations representing the Einstein gravitational field has made the finding of exact solutions an extremely difficult problem. The situation is particularly difficult when considering a metric that contains more than two functions, if, in particular, each of these is a function of more than any two of the coordinates. It is unfortunately the case, however, that the gaining of a complete picture of the non-Newtonian aspects of Einstein gravitation relies, to a certain extent at least, on finding such solutions. Thus, one might yield to the compromise of investigating those space-times that have somewhat special forms in the hope of generating field equations that might give exact solutions.

Weyl [1] gave a particularly simple form for the static axially symmetric metric which involved only two unknown functions of just two coordinates. The well-known Weyl formulation shows the existence of a coordinate system in which the line element has the form (with $c$ chosen as unity)

$$\text{ds}^2 = e^{2\psi} \text{dt}^2 - e^{2\gamma-2\psi} (\text{d}\rho^2 + \text{d}z^2) - \rho^2 e^{-2\psi} \text{d}\phi^2$$

(1)

where $\psi$ and $\gamma$ are functions of $\rho$ and $z$ only. The vacuum field equations, $R_{\mu\nu} = 0$, then impose the following conditions on $\psi$ and $\gamma$: 

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One of the first exact wave solutions of the field equations was given by Einstein and Rosen [2]. These authors treated a field of the Weyl type with the transformations \( z \rightarrow it \) and \( t \rightarrow iz \). In this case \( \psi \) and \( \gamma \) become functions of \( \rho \) and \( t \), while the line element and field equations transform into

\[
\begin{align*}
\rho \phi + (1/\rho) \rho \phi + \rho \phi_{zz} &= 0 \\
\rho \phi + (1/\rho) \rho \phi - \rho \phi_{t t} &= 0 \\
\rho \phi = \rho (\rho^2 - \rho z^2) \\
\rho \phi &= 2 \rho \rho \rho \rho z
\end{align*}
\]

The familiar linear equation obeyed by \( \psi \) in either case is easily solved and \( \gamma \) can then be obtained by straightforward integration. Complete solutions to these rather simple and special systems are thus readily found.

More recently, in discussing gravitational induction effects in axially symmetric fields, Levy [3] considered a “pseudo-Weyl metric” in which the functions \( \psi \) and \( \gamma \) appearing in (1) are taken as functions of \( \rho, z, \) and \( t \). These functions might be regarded as describing the field of an axially symmetric body that changes its shape in time, but in such a way as to retain its symmetry axis. The complete metric of Levy contains off-diagonal terms that are treated as small perturbations on the pseudo-Weyl field, while \( \psi (\rho, z, t) \) and \( \gamma (\rho, z, t) \) are assumed to coincide with the Weyl solutions \( \psi (\rho, z) \) and \( \gamma (\rho, z) \) for \( t \leq 0 \). In particular, Levy assumes that \( \psi \) has general solution, in spherical coordinates, of the form

\[
\psi (r, \theta, t) = \sum_{l=0}^{\infty} \left[ \frac{A_l(t)}{r^{l+1}} + B_l(t) r^l \right] P_l (\cos \theta)
\]

with the series coefficients taking on (nonzero) constant values for \( t \leq 0 \).

It is the purpose of this note to present a full discussion of the pseudo-Weyl vacuum field for the case in which \( \psi \) and \( \gamma \) depend on the three coordinates \( \rho, z, \) and \( t \). We show that no solutions (\( \psi \) and \( \gamma \)) exist that depend upon all three coordinate variables \( \rho, z, \) and \( t \); as might be expected then, no solutions exist for the “pseudo-Einstein-Rosen metric” which depend upon these three coordinates. (We are specifically addressing the case of variable solutions as opposed to any possible solutions that are simply constant functions of \( \rho, z, \) and \( t \).) Thus, the only solutions that exist for the metrics given by the forms (1) and (1a) are those given by Weyl and by Einstein and Rosen. This result might be regarded as a Birkhoff-type result for the diagonal, axially symmetric metric.