Spherically Symmetric Solutions in Five-Dimensional General Relativity

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Abstract

The most general time-independent spherically symmetric (in the usual three space dimensions) solution to the five-dimensional vacuum Einstein equations is found, subject to the existence of a Killing vector in the fifth direction. The significance of these solutions is discussed within the context of a previously proposed extension of the Kaluza-Klein model in which the universe, although (4 + 1)-dimensional, has evolved over cosmic times into an effectively (3 + 1)-dimensional one.

§(1): Introduction

The idea that the various forces of nature might be unified by enlarging the dimensionality of space-time has a long and generally honorable history that goes back to the work of Nordstrom in 1914 and Kaluza in 1921 [1, 2]. Its earlier adherents were mainly those interested in extending general relativity, while of late increased interest has been evident in the particle physics community, especially among those investigating extended supersymmetry [3-5]. Both the appeal and the frustration of this approach were touched on by Einstein and Pauli [6], who wrote in 1943:

When one tries to find a unified theory of the gravitational and electromagnetic fields, he cannot help feeling that there is some truth in Kaluza's five-dimensional theory. Yet its

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foundation is unsatisfactory insofar as, with respect to the group of admissible coordinate
transformations, the fifth, space-like coordinate is treated quite differently from the others.

Similar remarks, of course, apply whether one is interested in $4 + n$ dimensions
rather than five, or whether one seeks to include the strong and weak interactions in addition to electromagnetism.

In a previous paper [7], we argued that while the universe appears to be
$(3 + 1)$-dimensional, this appearance may be deceiving. We pointed out that
there is a simple solution to Einstein's equations (the Kasner solution) which
describes the evolution over cosmic times of a space-time with more than three
spatial dimensions, such that at the present epoch the extra dimensions have
shrunk to a size comparable to the Planck length. The specific example we chose
had one extra dimension, whose residual effects could be interpreted as the elec-
 tromagnetic interaction together with that of a scalar field. Other examples can
presumably be chosen to reproduce the effects of other, more complicated,
gauge theories.

In this paper, we shall eschew further cosmological speculation in favor of a
more detailed look at the properties of solutions to Einstein's equations on a
$(4 + 1)$-dimensional manifold. We shall demand the solutions be asymptotically
flat, which is inappropriate for an evolving universe, but which should be relevant
to describing our local environment.

Thus, we are chiefly interested in the vacuum Einstein equations

$$R_{\mu\nu} = 0$$  \hspace{1cm} (1)

in $(4 + 1)$-dimensions. When projected down to an effective $(3 + 1)$-dimensional
manifold, the degrees of freedom in $g_{\mu\nu}$ ($\mu, \nu = 0, 1, 2, 3, 5$) represent the gravi-
tational and electromagnetic fields, and an extra scalar field. For most of this
paper, we shall assume that the metric possesses a spacelike Killing vector $\xi^a$
which can be taken to be

$$\xi^a \frac{\partial}{\partial x^a} = \frac{\partial}{\partial x^5}$$  \hspace{1cm} (2)

We are not forced to demand the existence of this Killing vector, but since the
fifth-dimension is so small, it is intuitively reasonable that the gross features of
matter can be well described by ignoring any dependence on $x^5$. (From a quan-
tum mechanical viewpoint, the uncertainty principle tells us that the energy re-
quired to produce excitations in the fifth direction is of the order of $10^{20}$ Mev.)
Furthermore, the existence of $\xi$ provides an elegant and unambiguous way to
project out the fifth dimension. The formalism for doing this is developed in
Section 2.

In Section 3, we find the most general time-independent spherically sym-
metric (in the usual three space dimensions) solutions to equation (1). These are
characterized by three real parameters. In Section 4, we probe equation (1) by