Structure of the Generalized Friedmann Problem

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An investigation of those cases of the generalized Friedmann equation which are solvable in terms of elementary or elliptic functions is undertaken together with a study of the time gauges which allow this to occur. This is accomplished by examining the natural choices of independent and dependent variables in this problem using manipulations like those of the Kepler problem, which is shown to be equivalent to a generalized Friedmann problem, thus clarifying the similarities between the simplest solutions of each.

1. INTRODUCTION

This article is the first in a series dealing with the choice of time in gravitational dynamics. In gravitational dynamics time is interesting not only from a conceptual point of view, but also as a technical tool. For example, in Newtonian mechanics time is considered to be "absolute" yet it also plays a less lofty role in particle dynamics as a parameter for individual orbits. In this latter context it is often convenient to use another parameter, or another "time choice", which in the Kepler problem is most naturally taken to be an angular variable.

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In general relativity time holds a less prominent position than in Newtonian mechanics. While in Newtonian mechanics time and space are separated, and the geometry of space is fixed, in general relativity they merge into a single object, spacetime, where the geometry must now satisfy Einstein's field equations. On the one hand time is demoted in conceptual importance since there is in general no preferred choice of time, but on the other hand exploiting this freedom of choice now plays a crucial role in analyzing the field equations.

In general relativity "time" can either be considered from a local point of view where it is independently measured at different points of "space" or from a nonlocal point of view involving the correlation or some kind of synchronization of such local times all over space. The first point of view is realized by representing physical quantities in terms of a timelike congruence of test observers in the spacetime, most naturally called the congruence point of view. The second point of view is realized instead by referring physical quantities to an integrable family of such test observers whose worldlines are orthogonal to a slicing or foliation of the spacetime by a family of spacelike hypersurfaces, which will be called the hypersurface point of view. Each of these represents a partial splitting of spacetime, splitting off only the time and space respectively, leaving a gaugelike freedom to define the space and time respectively.

In this series of articles the hypersurface point of view will be assumed and spacetime (or at least a part of spacetime) will be considered to be foliated by a set of spacelike hypersurfaces. These can then be threaded by an arbitrary but transversal congruence which serves only to identify points of "space" at different times, leading to the complete splitting of spacetime into "space plus time" and a point of view often referred to as the 3+1 approach [1]. For a given choice of slicing, the remaining freedom in the choice of threading is the spatial gauge freedom, which also plays an important role in gravitational dynamics.

There are many examples in relativity in which the choice of slicing is important. In general, for example, the study of the initial value problem motivates a slicing by a family of spacelike hypersurfaces of constant mean extrinsic curvature, since that problem decouples on a hypersurface of this type. Originally suggested by York [2] in this context, it has proven important in geometrically slicing spacetimes [3] even in numerical relativity. The synchronous time gauge [4], which fixes both the slicing and the parametrization, has also played a key role in attempts to understand properties of spacetime singularities.

Examples of specific classes of spatially inhomogeneous spacetimes where the choice of slicing is important may also be cited. For example,