Properties of the Set of Nonreflecting Points

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Abstract

We show that the set of nonreflecting points C is without interior in the space-time. We also construct a space-time in which C is dense in some open subset of M.

§(1): Introduction and Preliminaries

The purpose of this paper is to examine the structure of the set of points for which the reflecting property fails. The reflecting space-time was investigated by Hawking and Sachs [1], and the properties of causal structure of this space-time were fully discussed by Budic and Sachs [2]. The motivation of the investigation of such space-time is in the following argument. If the space-time is not past-reflecting, some observer receives completely unpredictable information from a hidden region of space-time; such behavior may be inappropriate in doing physics. If we accept past-reflectivity and if the space-time is time-symmetric, it is future and past-reflecting. We can show that if the space-time is not past-reflecting, it may be pathological, that is, it may be the space-time in which we receive new information from the indefinitely large regions all the time of some time interval.

We define a space-time \((M, g)\) to be a pair of manifolds \(M\) and a Riemannian metric with signature \(-2\). We assume that the base manifold \(M\) is Hausdorff, paracompact, and connected.

We shall consider in this paper the distinguishing space-time, i.e., the space-time in which either \(I^+(x) = I^+(y)\) or \(I^-(x) = I^-(y)\) implies \(x = y\). We use the same sign conventions as used in Hawking and Ellis [3].

A space-time is said to be past-reflecting if the following condition holds:
For all events $x$ and $y$ in $M$, $I^+(y) \supset I^+(x)$ implies $I^-(x) \supset I^-(y)$; it is said to be future-reflecting if $I^-(x) \supset I^-(y)$ implies $I^+(y) \supset I^+(x)$. We define the following sets as they were defined by Hawking and Sachs [1]. Let $U$ be an open set; then the chronological common past $\downarrow U$ is the largest open set each event of which can send a message to each observer in $U$, i.e., $\downarrow U \equiv \text{Interior} \{m \in M \mid \text{for all } u \in U \text{ there is a future-directed timelike curve from } m \text{ to } u\}$. $\uparrow U$ is defined analogously. It is easy to show that past-reflecting $\iff I+(x) = I^-(x)$ for all $x \in M$, and its dual [1].

§(2): The Set of Nonreflecting Points

Let $C^-$ be the set of points $x$ where $I^+(x) \neq I^-(x)$, and let $C^+$ be the analogously defined set. First we show the following lemma.

Lemma. Let $\gamma$ be any timelike curve, then $\gamma \cap C^-$ and $\gamma \cap C^+$ are countable subsets of $M$.

Proof. From the definition $x \in C^-$ iff $\downarrow I^+(x) \setminus I^-(x) (\equiv A(x)) \neq \emptyset$. Let $x$ and $\tilde{x}$ be two different points of $\gamma \cap C^-$; then $A(x) \cap A(\tilde{x}) = \emptyset$, because either $x << \tilde{x}$ or $\tilde{x} << x$ holds ($\gamma$ is timelike). If $\gamma \cap C^-$ is uncountable, then we obtain an uncountable collection of nonempty nonintersecting open sets in the space-time. This result contradicts the paracompactness of the space-time. Similarly we can show that $\gamma \cap C^+$ is countable.

As a simple consequence of the lemma, we obtain the following theorem.

Theorem. The set $C^- \cup C^+$ is a subset of $M$ without interior.

§(3): Construction of a Model with Dense $C^-$

In the previous section we proved that $C^-$ is the subset of $M$ without interior. In this section we want to construct a space-time in which $C^-$ is dense in some open subset of $M$ (see Fig. 1).

One may think that the connected component of $x \in C^-$ in some neighborhood has a suitable structure. However, this set also has very complicated structure. For example, $C^-$ does not have a manifold structure, as the following example demonstrates (at point $P \in C^-$ the manifold structure cannot be defined).

Let $M$ be a manifold obtained from $R^3$ (with usual Euclidean coordinates) by suppressing the subset $\{x_0 = 0\} \cap \{x_2 \leq -\frac{1}{2}\} \cap \{(x_1)^2 + (x_2)^2 \leq 1\}$, and let the metric $g$ be a usual Lorentz metric on $M$ such that with respect to $g$, the $x_0$-axis is timelike. In this space-time $(M, g)$, the set $C^-$ is the following one: Suppose $P = (1, 0, 0)$; then

$$C^- = C_-^1 \cup C_-^2 \cup \{P\}$$

where

$$C_-^1 = E^-(P) \cap \{x_0 > 2x_2 + 1\} \cap \{x_0 > 0\}$$