Null Cut Loci of Spacelike Surfaces

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The null cut locus of a spacelike submanifold of codimension 2 in a space-time is defined. In globally hyperbolic space-times, it is shown that the future (past) null cut locus $C^+(H) [C^-(H)]$ of a compact, acausal, spacelike submanifold $H$ of codimension 2 is a closed subset of the space-time, and each point $x \in C^+(H)$ is either a focal point of $H$ along some future-directed null geodesic meeting $H$ orthogonally or there exist at least two null geodesics from $H$ to $x$, realizing the distance between $H$ and $x$ or both. Also, it can be shown that the assumptions of the Penrose's singularity theorem for open globally hyperbolic space-times may be weakened to the space-times which are conformal to an open subset of an open globally hyperbolic space-time.

1. INTRODUCTION

The future cut point of a future-directed null geodesic $\gamma : [0, a) \rightarrow M$ in a space-time $M$ is defined to be the point $q = \gamma(t_0)$, where $t_0 = \sup \{ t \in [0, a) | d[\gamma(0), \gamma(t)] = 0 \}$, with $0 < t_0 < a$ and $d$ is the Lorentzian distance function. Then, the future null cut locus $C^+(p)$ of a point $p \in M$ is defined to be the set of future null cut points of all future inextendable null geodesics $\gamma : [0, a) \rightarrow M$, with $\gamma(0) = p$. In globally hyperbolic space-times, it can be shown that $C^+(p)$ is a closed subset of the space-time, and each $x \in C^+(p)$ is either the first conjugate point of $p$ along a future-directed null geodesic from $p$ to $q$ or there exist at least two null geodesics from $p$ to $q$ realizing the distance $d(p, q) = 0$ or both. Furthermore, a singularity

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theorem is proved by making use of the invariance of null cut points under conformal imbeddings (cf. [1], pp. 230–237).

In this paper, we generalize the above concept to the future null locus of a spacelike submanifold $H$ of codimension 2 in an $n$-dimensional spacetime $M$, where $n \geq 3$. In our definition of the future null cut locus $C^+_N(H)$ of $H$, $C^+_N(H)$ corresponds to the set points of the future horismos $E^+(H) = J^+(H) - I^+(H)$ of $H$ where the nontrivial null generators of $E^+(H)$ leave $E^+(H)$ into the future, where $J^+(H)$ and $I^+(H)$ are, respectively, the causal and chronological future of $H$. Then, by making use of the main ideas in [1], we show that, in globally hyperbolic space-times, the future null cut locus of a compact, acausal, spacelike submanifold of codimension 2 has the properties similar to the future null cut locus of a point. Then, by making use of the properties of the null cut loci of spacelike submanifolds of codimension 2, we show that the normal bundle of a connected, compact, acausal, spacelike submanifold $H$ of codimension 2 is trivial and, therefore, $H$ is orientable. Finally, we show that the assumptions of the Penrose's singularity theorem for open globally hyperbolic space-times (cf. [2], p. 263) can be weakened to the space-times which can be conformally imbedded into an open globally hyperbolic space-time using the invariance of the null cut loci of spacelike submanifolds of codimension 2 under conformal imbeddings.

2. PRELIMINARIES

A space-time $(M, g)$ is an $n$-dimensional connected, oriented, time-oriented Lorentzian manifold with metric $g$ of signature $(-+++\cdots+)$. A spacelike surface is a connected, imbedded, spacelike (smooth) submanifold of codimension 2 in $M$. (Note that the normal bundle $H^\perp$ of a spacelike surface $H$ has 2-dimensional timelike fibers, each of which contains two well-defined null directions). The Lorentzian distance between a set $H$ and a point $p$ in a space-time $M$ is defined to be

$$d(H, p) = \begin{cases} \sup \{L(y) \mid y \in \Omega_{H,p} \} & \text{if } \Omega_{H,p} \neq \emptyset \\ 0 & \text{if } \Omega_{H,p} = \emptyset \end{cases}$$

where $\Omega_{H,p}$ is the set of all future-directed piecewise differentiable non-spacelike curves from $H$ to $p$, and $L(y)$ is the length of $y \in \Omega_{H,p}$. 

3. FUTURE NULL CUT LOCUS OF A SPACELIKE SURFACE

**Definition 1.** Let $H$ be a spacelike surface in a space-time $M$ and $\gamma : [0, a) \to M$ be a future-directed null geodesic with $\gamma(0) \in H$, $\dot{\gamma}(0) \perp H$. A